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Self-Inductance of Solenoids, Bi-Dimensional Rings and Coaxial Cables

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Abstract. We compare the self-inductance formulae of Neumann, Weber, Maxwell and Graneau. To this end we present exact and algebraic formulae for the self-inductance of solenoids, bi-dimensional rings and coaxial cables. We show that these four formulas agree exactly with one another for closed circuits.

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1 Introduction

We shall utilize in this work a powerful method of calculating inductances. With this method one can obtain exact and algebraic results, instead of approximation formulae that are presented is most situations. We have recently presented this method[1]. Although Sommerfeld had presented a similar formula in his book ([2], p. 105), he dealt only with Neumann's expression. In this work, and in the preceding one [1], we extend the method for the inductance formulae of Weber, Maxwell and Graneau. Let us first discuss briefly their historical appearance.

Consider a frame of reference S with origin O and two current elements $I_i d\vec{\ell}_i$ and $I_j d\vec{\ell}_j$

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located relative to S at $\vec{r}_i$ and $\vec{r}_j$, respectively. In 1826 Ampère obtained the force exerted by j on i, $d^2\vec{F}_{ji}$, as ([3], Chapter 4):

$$d^2\vec{F}_{ji} = -\frac{\mu_o}{4\pi} I_i I_j \frac{\hat{r}_{ij}}{r_{ij}^2} \left[2(d\vec{\ell}_i \cdot d\vec{\ell}_j) - 3(\hat{r}_{ij} \cdot d\vec{\ell}_i)(\hat{r}_{ij} \cdot d\vec{\ell}_j) \right] \ , \tag{1.1}$$

where $\mu_o = 4\pi \times 10^{-7} kgmC^{-2}$ is the vacuum permeability, $r_{ij} = |\vec{r}_i - \vec{r}_j|$ and $\hat{r}_{ij} = (\vec{r}_i - \vec{r}_j)/r_{ij}$.

When we integrate this expression over the two closed circuits C_i and C_j the force can be written as:

$$\vec{F}_{C_jC_i} = \frac{\mu_o}{4\pi} I_i I_j \oint_{C_i} \oint_{C_j} \frac{d\vec{\ell}_i \times (d\vec{\ell}_j \times \hat{r}_{ij})}{r_{ij}^2} = -\frac{\mu_o}{4\pi} I_i I_j \oint_{C_i} \oint_{C_j} \frac{\hat{r}_{ij}}{r_{ij}^2} d\vec{\ell}_i \cdot d\vec{\ell}_j \ . \tag{1.2}$$

In 1845 F. Neumann introduced the coefficient of mutual inductance M^N showing that this force between two rigid closed circuits might be written as $I_i I_j \nabla M^N$, where

$$M^N = \frac{\mu_o}{4\pi} \oint_{C_i} \oint_{C_j} \frac{d\vec{\ell}_i \cdot d\vec{\ell}_j}{r_{ij}} \ . \tag{1.3}$$

In 1846 W. Weber introduced a force law from which he could derive as special cases Coulomb's force and Ampère's force (1.1), [3], Chapter 3. In 1848 he introduced a potential energy d^2U^W between two point charges dq_i and dq_j from which he could derive his force as

$$d^2U_{ij}^W = \frac{dq_i dq_j}{4\pi\varepsilon_o} \frac{1}{r_{ij}} \left(1 - \frac{\dot{r}_{ij}^2}{2c^2} \right) \ , \tag{1.4}$$

where $\varepsilon_o = 8.85 \times 10^{-12} C^2N^{-1}m^{-2}$ is the permittivity of free space, $c = 1/\sqrt{\mu_o\varepsilon_o} = 3 \times 10^8 ms^{-1}$ and $\dot{r}_{ij} = dr_{ij}/dt$.

Considering the neutral current elements as being composed of positive and negative charges ($dq_{-i} = -dq_{+i}$ and $dq_{-j} = -dq_{+j}$) and adding the energy of interaction between the positive and negative charges of one current element interacting with the positive and negative charges of the other current element yields: $d^2U_{ij}^W = I_i I_j d^2 M_{ij}^W$, where

$$d^2M_{ij}^W = \frac{\mu_o}{4\pi} \frac{(\hat{r}_{ij} \cdot d\vec{\ell}_i)(\hat{r}_{ij} \cdot d\vec{\ell}_j)}{r_{ij}} \ . \tag{1.5}$$

Here it was utilized $I_i d\vec{\ell}_i = dq_{+i}(\vec{v}_{+i} - \vec{v}_{-i})$ and $I_j d\vec{\ell}_j = dq_{+j}(\vec{v}_{+j} - \vec{v}_{-j})$, where $\vec{v}_a$ is the velocity of the charge dq_a relative to S, see [3], Sections 4.2 and 4.6.

Maxwell worked with an expression for M which was half Neumann's expression plus half Weber's expression. Nowadays the simplest way to derive Maxwell's formula is to work with

Darwin's lagrangian. Accordingly the energy of interaction between the charges dq_i and dq_j moving with velocities $\vec{v}_i$ and $\vec{v}_j$ is given by ([3], Section 6.8; [4], Section 12.7, pp. 593-595):

$$d^2U = \frac{dq_i dq_j}{4\pi\varepsilon_o}\frac{1}{r_{ij}}\left[1 - \frac{\vec{v}_i \cdot \vec{v}_j + (\vec{v}_i \cdot \hat{r}_{ij})(\vec{v}_j \cdot \hat{r}_{ij})}{2c^2}\right] . \tag{1.6}$$

Adding this expression for the positive and negative charge of one current element interacting with the positive and negative charge of the other current element as we did for Weber's law yields Maxwell's expression $d^2U_{ij}^M = I_i I_j d^2 M_{ij}^M$, where

$$d^2M_{ij}^M = \frac{\mu_o}{4\pi}\frac{1}{2}\left[\frac{d\vec{\ell}_i \cdot d\vec{\ell}_j}{r_{ij}} + \frac{(\hat{r}_{ij} \cdot d\vec{\ell}_i)(\hat{r}_{ij} \cdot d\vec{\ell}_j)}{r_{ij}}\right] . \tag{1.7}$$

More recently P. Graneau introduced a fourth formula to calculate the mutual energy or mutual inductance between two current elements from which he could derive directly Ampère's force (1.1), namely ([5], p. 212):

$$d^2M_{ij}^G = \frac{\mu_o}{4\pi}\left[3\frac{(\hat{r}_{ij} \cdot d\vec{\ell}_i)(\hat{r}_{ij} \cdot d\vec{\ell}_j)}{r_{ij}} - 2\frac{d\vec{\ell}_i \cdot d\vec{\ell}_j}{r_{ij}}\right] . \tag{1.8}$$

All these four expressions for d^2M can be summarized in a single formula, namely:

$$d^2M_{ij} = \frac{\mu_o}{4\pi}\left[\left(\frac{1+k}{2}\right)\frac{d\vec{\ell}_i \cdot d\vec{\ell}_j}{r_{ij}} + \left(\frac{1-k}{2}\right)\frac{(\hat{r}_{ij} \cdot d\vec{\ell}_i)(\hat{r}_{ij} \cdot d\vec{\ell}_j)}{r_{ij}}\right] , \tag{1.9}$$

where if $k = 1,\ -1, 0$ or $-\ 5$ we obtain, respectively, the formulas of Neumann, Weber, Maxwell and Graneau.

It has been known for a long time that all these formulas agree with one another when we calculate the mutual inductance between any two closed circuits. Only recently we have been able to prove that the same is also valid for the self-inductance of a single closed circuit of arbitrary form, [6]. In this work we illustrate this equivalence calculating exactly with the four formulas presented above the self-inductance of a solenoid and bi-dimensional ring, as this detailed comparison had never been done before.

For filiform circuits the integration of Eq. (1.9) yields infinite results. To avoid this we generalized this expression for current flowing over the surface of bi-dimensional conductors, namely ([1]):

$$d^4M_{ij} = \frac{\mu_0}{4\pi}\frac{1}{\omega_i\omega_j}\left[\left(\frac{1+k}{2}\right)\frac{(\hat{\ell}_i \cdot \hat{\ell}_j)}{r_{ij}} + \left(\frac{1-k}{2}\right)\frac{(\hat{r}_{ij} \cdot \hat{\ell}_i)(\hat{r}_{ij} \cdot \hat{\ell}_j)}{r_{ij}}\right]da_i da_j, \tag{1.10}$$

where $\hat{\ell}$ is the unit vector indicating the direction of the current flow, ω is the width (transverse to $\hat{\ell}$) of the conductor and da is an element of area in the conductor (see Figure 1 for an example).

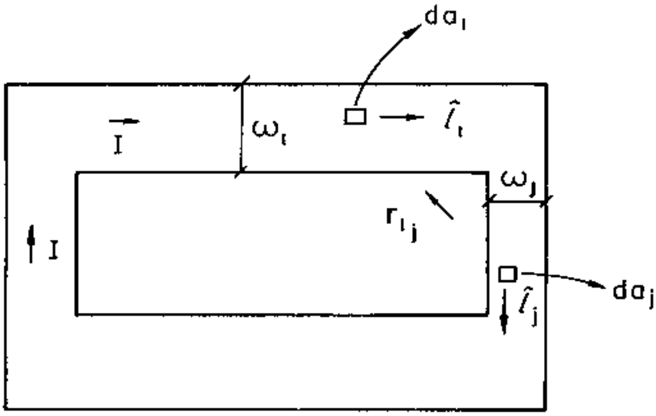

Figure 1: Bi-dimensional circuit illustrating the meaning of ω, $\hat{\ell}$ and da.

2 Solenoids and Bi-Dimensional Rings

The self-inductance of the solenoid and of the ring will be calculated with the geometry presented in Fig. 2. The cylinder has a length ℓ and radius a, in which flows an uniform surface poloidal current density $\vec{K}$ given by $(I/\ell)\hat{\phi}$, where $\hat{\phi}$ is the unit vector in cylindrical coordinates (ρ, ϕ, z). Here I is the total current flowing through the length ℓ.

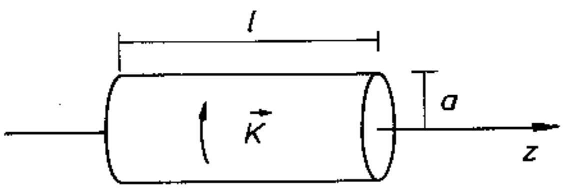

Figure 2: Cylinder with surface poloidal current density.

On replacing in Eq. (1.10): $\hat{\ell}_i = \hat{\phi}_i$, $\hat{\ell}_j = \hat{\phi}_j$, $da_i = adz_i d\phi_i$, $da_j = adz_j d\phi_j$, $\omega_i = \omega_j = \ell$, $\vec{r}_i = a\hat{\rho}_i + z_i\hat{z}$, $\vec{r}_j = a\hat{\rho}_j + z_j\hat{z}$ and the limits of integration yields

$$L_{poloidal} = \frac{\mu_0}{4\pi}\frac{a^2}{\ell^2}\int_0^{2\pi} d\phi_i \int_0^{2\pi} d\phi_j \int_0^{\ell} dz_i \int_0^{\ell} dz_j$$

$$
\begin{aligned}
&\times \left[\left(\frac{1+k}{2}\right)\frac{\cos(\phi_i-\phi_j)}{[2a^2(1-\cos(\phi_i-\phi_j))+(z_i-z_j)^2]^{1/2}}\right. \\
&\left.+ \left(\frac{1-k}{2}\right)\frac{a^2\sin^2(\phi_i-\phi_j)}{[2a^2(1-\cos(\phi_i-\phi_j))+(z_i-z_j)^2]^{3/2}}\right] \\
= \ &\frac{2\mu_0 a}{3}\left[\frac{1}{q}(\mathbf{K}(q)-\mathbf{E}(q))+p^2\left(\frac{\mathbf{E}(q)}{q}-1\right)\right], \qquad (2.1)
\end{aligned}
$$

where $p \equiv 2a/\ell$, $q \equiv p/(1+p^2)^{1/2}$, $\mathbf{K}$ and $\mathbf{E}$ are, respectively, the complete elliptic integrals of the first and second kinds[7], pp. 907–908. The first to obtain the self-inductance with this geometry in terms of elliptic integrals was Lorenz [8], p. 142. He worked only with Neumann's formula. Here we obtained for the first time in the literature the same result with the other formulae. This is a highly non trivial result.

The result in (2.1) is independent of k, so it has the same value for the formulae of Neumann, Weber, Maxwell and Graneau. It is also exact and presented as an analytically simple expression. As it was obtained without restrictions on ℓ and a, it is valid either for the self-inductance of a long solenoid of length ℓ and radius a ($\ell \gg a$), obtained by winding N turns of wire on a cylindrical form, or for the self-inductance of a bi-dimensional ring ($\ell \ll a$).

The expansions of Eq. (2.1) for the two limits cited above ($\ell \gg a$ and $\ell \ll a$) are, respectively:

$$L_{solenoid} \approx \frac{\mu_0 \pi a^2}{\ell}\left(1 - \frac{8}{3\pi}\frac{a}{\ell} + \frac{1}{2}\frac{a^2}{\ell^2}\right), \qquad (2.2)$$

$$L_{ring} \approx \mu_0 a\left(\ln\left(\frac{8a}{\ell}\right) - \frac{1}{2}\right). \qquad (2.3)$$

In most textbooks we find a result for the solenoid with N turns valid for $\ell \gg a$ (see, for instance, [9], p. 442). The method utilized in the textbooks is given by $L = d\Phi/dI_1$, where Φ is the magnetic flux over the circuit, and I_1 is the current in each turn. This method is only useful in highly symmetrical situations in which we can easily calculate Φ. The result they obtain is given by

$$L_{poloidal}^{textbooks} = \mu_0 \pi N^2 \frac{a^2}{\ell}. \qquad (2.4)$$

Eq. (2.2) presents this result with corrections of higher orders.

The difference in the factor N^2 is only a matter of definition. In the textbooks the magnetic energy of this system is given by $LI_1^2/2$, with L given by (2.4), as they concentrate their analysis in the current I_1 in each turn. If we concentrate on the total current $I = NI_1$ over the whole length ℓ of the cylinder, the magnetic energy will be given by $LI^2/2$, with L

given by (2.2), so that the measurable self energy agrees with the previous value. However, this last approach is preferable in some respects as it preserves the idea of L depending only on the geometry of the system. In the solenoid when we change the number of turns N, keeping I_1 constant, the geometry (length ℓ and radius a of the cylinder) is not modified, so that L should remain the same. This happens with (2.2) but not with (2.4).

3 Coaxial Cable

In Fig. 3 we present the geometry for calculating the self-inductance of the coaxial cable. There are two coaxial cylinders of radius a and b, and length ℓ. The surface current density $\vec{K}$ flows uniformly along the $\hat{z}$ direction on the outer cylinder and $-\hat{z}$ on the inner one.

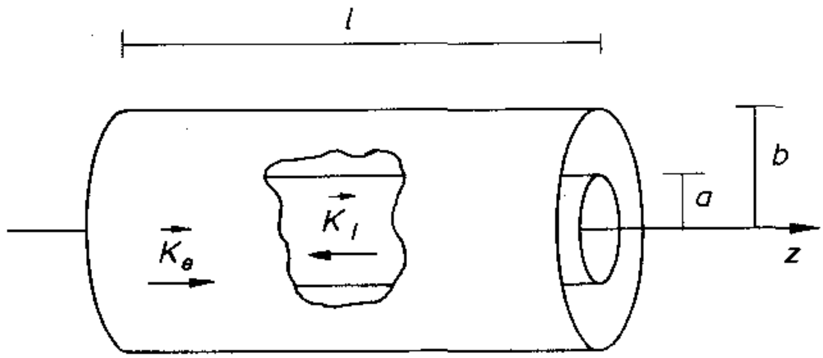

Figure 3: Two concentrical cylinders making a coaxial cable, with opposite currents flowing along the axial direction.

The self-inductance of the coaxial cable is given by: $L_a + L_b + 2M_{ab}$. Here L_a (L_b) is the self-inductance of the cylinder with radius a (b), and M_{ab} is the mutual inductance between the two cylinders. For L_a we substitute in (1.10): $\hat{\ell}_i = \hat{\ell}_j = \hat{z}$, $da_i = adz_i d\phi_i$, $da_j = adz_j d\phi_j$, $\omega_i = \omega_j = 2\pi a$, $\vec{r}_i = a\hat{\rho}_i + z_i\hat{z}$, $\vec{r}_j = a\hat{\rho}_j + z_j\hat{z}$ and the limits of integration to obtain:

$$
\begin{aligned}
L_a &= \frac{\mu_0}{16\pi^3}\int_0^{2\pi} d\phi_i \int_0^{2\pi} d\phi_j \int_0^{\ell} dz_i \int_0^{\ell} dz_j \\
&\times \left[\left(\frac{1+k}{2}\right)\frac{1}{[2a^2(1-\cos(\phi_i-\phi_j))+(z_i-z_j)^2]^{1/2}}\right. \\
&+ \left.\left(\frac{1-k}{2}\right)\frac{(z_i-z_j)^2}{[2a^2(1-\cos(\phi_i-\phi_j))+(z_i-z_j)^2]^{3/2}}\right]. \qquad (3.1)
\end{aligned}
$$

For the coaxial cable we are just interested in the result of Eq. (3.1) for the limit $\ell \gg a$. Considering this approximation we obtain:

$$L_a \approx \frac{\mu_0 \ell}{2\pi}\left[\ln\left(\frac{2\ell}{a}\right)+\left(\frac{k-3}{2}\right)+\frac{2}{\pi}\frac{a}{\ell}(3-k)+\frac{1}{2}\frac{a^2}{\ell^2}(k-2)\right]. \tag{3.2}$$

Analogously:

$$L_b \approx \frac{\mu_0 \ell}{2\pi}\left[\ln\left(\frac{2\ell}{b}\right)+\left(\frac{k-3}{2}\right)+\frac{2}{\pi}\frac{b}{\ell}(3-k)+\frac{1}{2}\frac{b^2}{\ell^2}(k-2)\right]. \tag{3.3}$$

For calculating the mutual inductance between the two cylinders of Fig. 3 we substitute in (1.10): $\hat{\ell}_i = -\hat{\ell}_j = \hat{z}$, $da_i = adz_i d\phi_i$, $da_j = bdz_j d\phi_j$, $\omega_i = 2\pi a$, $\omega_j = 2\pi b$, $\vec{r}_i = a\hat{\rho}_i + z_i\hat{z}$, $\vec{r}_j = b\hat{\rho}_j + z_j\hat{z}$ and the limits of integration:

$$\begin{aligned} M_{ab} &= -\frac{\mu_0}{16\pi^3}\int_0^{2\pi} d\phi_i \int_0^{2\pi} d\phi_j \int_0^{\ell} dz_i \int_0^{\ell} dz_j \\ &\quad \times \left[\left(\frac{1+k}{2}\right)\frac{1}{[a^2+b^2-2ab\cos(\phi_i-\phi_j)+(z_i-z_j)^2]^{1/2}}\right. \\ &\quad + \left.\left(\frac{1-k}{2}\right)\frac{(z_i-z_j)^2}{[a^2+b^2-2ab\cos(\phi_i-\phi_j)+(z_i-z_j)^2]^{3/2}}\right] \\ &\approx -\frac{\mu_0}{2\pi}\left[\ln\left(\frac{2\ell}{a}\right)-\ln r+\left(\frac{k-3}{2}\right)+\frac{(3-k)}{\pi}\frac{a}{\ell}|1-r|\mathbf{E}\left(\frac{2i\sqrt{r}}{|1-r|}\right)\right. \\ &\quad + \left.\frac{(k-2)}{4}(1+r^2)\frac{a^2}{\ell^2}\right], \end{aligned} \tag{3.4}$$

where $i = \sqrt{-1}$ is the imaginary unit, $r \equiv b/a > 1$ and we have considered $\ell \gg b > a$.

Finally, as $L_{coaxial} = L_a + L_b + 2M_{ab}$, from (3.2) to (3.4) we obtain:

$$L_{coaxial} \approx \frac{\mu_0 \ell}{2\pi}\left[\ln r + \frac{2}{\pi}\frac{a}{\ell}(3-k)\left(1+r-|1-r|\mathbf{E}\left(\frac{2i\sqrt{r}}{|1-r|}\right)\right)\right]. \tag{3.5}$$

In [10], Vol. 2, pp. 24–1 to 24–3, we find the self-inductance of a coaxial cable analogous to that of Fig. 3. It was obtained utilizing $U = LI^2/2$, where U is the magnetic energy calculated through $\int\int\int B^2 dV/(2\mu_0)$ (B being the magnitude of the magnetic field). The result they obtained (supposing $\ell \gg b > a$) was:

$$L_{coaxial}^{textbooks} = \frac{\mu_0 \ell}{2\pi}\ \ln r. \tag{3.6}$$

This result is exactly the zeroth approximation order of Eq. (3.5).

4 Conclusions

In this work we have obtained analytically exact expression for the self-inductance of a solenoid or a bi-dimensional ring, Fig. 2 and Eq. (2.1), using a powerful method of inductance calculation [1]. With this method we have also calculated the self-inductance for the coaxial cable, Fig. 3, in the limit of its length being much greater than its outer radius, Eq. (3.5).

For the cylinder with closed poloidal lines of current, Fig. 2, we have obtained an exact equivalence between the formulae of Neumann, Weber, Maxwell and Graneau, see Eq. (2.1). This exact equivalence is the main result of this paper.

On the other hand, for the cylinders with open axial lines of current, Fig. 3, we have not obtained this equivalence as the final expression depends on k, see Eqs. (3.2) to (3.5). This dependence on k will disappear if we consider closed lines of current (taking into account, for instance, the radial currents at the lids in the two extremities of the coaxial cable of Fig. 3) [6]. This means that this dependence on k is not important for any experiment involving only closed circuits as it will disappear and will not be detected by any experimental means.

For a general proof that the self inductance of a closed circuit of arbitrary form is the same with all these expressions, see [6]. In this work we have been concerned in showing this complete equivalence in specific examples which allowed exact integrations, as was the case of the solenoid and bi-dimensional ring.

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