Equivalence Between Ampère and Grassmann's Forces

A. K. T. Assis and Marcelo A. Bueno

***Abstract*—We calculate the force on part of a single closed circuit due to the remaining circuit in four different geometries according to the forces of Ampère and Grassmann. All analytical calculations are performed using surface or volume current elements in order to avoid the divergences which appear with linear current elements of zero diameter. We conclude that when we consider the action of a closed circuit as a whole and utilize only circuits with closed lines of current, there will be an equivalence between the expressions of Grassmann and Ampère. This means that both of them are compatible with the experimental findings related to Ampère's bridge, contrary to the opinion of some authors.**

I. Introduction

When an electric current flows in two metallic circuits, there is a ponderomotive force between them. This has been known since 1820. There are two main expressions from which this force can be calculated: the forces of Ampère and Grassmann. According to Ampère, the force $d^2\vec{F}_{ji}$ exerted by a current element $I_j d\vec{l}_j$, located at $\vec{r}_j$, on another current element $I_i d\vec{l}_i$, located at $\vec{r}_i$, is given by [1]

$$d^2\vec{F}_{ji}^A = -\frac{\mu_0}{4\pi} I_i I_j \frac{\hat{r}}{r^2} [2(d\vec{l}_i \cdot d\vec{l}_j) - 3(\hat{r} \cdot d\vec{l}_i)(\hat{r} \cdot d\vec{l}_j)] \tag{1}$$

where $\mu_0 = 4\pi \times 10^{-7}$ kgm/C^2 is the vacuum permeability, $r \equiv |\vec{r}_i - \vec{r}_j|$ is the distance between the elements, and $\hat{r} \equiv (\vec{r}_i - \vec{r}_j)/r$ is the unit vector pointing from j to i.

On the other hand, Grassmann's force (based on Biot-Savart's magnetic field) states that the force of j on i is given by [1], [2]

$$\begin{aligned} d^2\vec{F}_{ji}^G &= I_i d\vec{l}_i \times d\vec{B}_j^{B-S} = I_i d\vec{l}_i \times \left(\frac{\mu_0}{4\pi} I_j \frac{d\vec{l}_j \times \hat{r}}{r^2}\right) \\ &= -\frac{\mu_0}{4\pi} \frac{I_i I_j}{r^2} [(d\vec{l}_i \cdot d\vec{l}_j)\hat{r} - (d\vec{l}_i \cdot \hat{r}) d\vec{l}_j]. \end{aligned} \tag{2}$$

Ampère's force (1) follows Newton's third law (action and reaction) in the strong form as the force is always directed along the line joining the elements. Grassmann's force between current elements (2) does not obey Newton's third law, with the exception of some particular situations. This is not important, since current elements cannot exist in practice. We can only measure forces between closed circuits, or complementary parts of a closed circuit. It has been known that the integrated force of a closed circuit on a current element of another circuit has the same value according to both expressions [3]. This means that Grassmann's force will also follow the action and reaction principle when applied to closed circuits. Moreover, this fact indicates that the two expressions cannot be distinguished in experiments involving two or more closed circuits.

Recently, many experiments have been performed with a single closed circuit trying to distinguish between these two expressions [4]–[9]. The idea is to measure the force on part of a circuit due to the remaining parts of the circuit. Experimentally, this can be done by connecting the two metallic parts by liquid mercury so that the ponderomotive force on part of a circuit can be measured without interrupting the current. Although most experiments seem to favor Ampère's force against Grassmann's one, this is still a controversial subject [10]–[12].

If we approach this subject theoretically, we face problems of divergence when trying to integrate (1) or (2) for a single circuit. In order to avoid this divergence, we can either utilize numerical integration using current elements of finite size (typically of the order of the interatomic lattice spacing) [8], [10], [13], [14] or we can perform analytical integrations using surface or volume current elements [15]. In this work, we follow this last approach.

II. Circuits with Surface Current Elements

The first geometry we consider is that of Fig. 1. We have a circuit with surface current elements. We divide this circuit in 6 pieces and we suppose that the constant current in these pieces flows uniformly over their cross-sections. This means that the current in the whole piece 1 (the inferior rectangle with sides ℓ_3 and w) is supposed to be constant over its cross section and flowing along the positive x direction, and similarly for the other pieces. The bridge (B) is supposed to consist of pieces 3, 4, and 5.[1] We call support (S) the remaining pieces 6, 1, and 2.

[1]The name bridge has been utilized in conformity with the famous Ampère's bridge or hairpin experiment [8].

Manuscript received October 1, 1994; revised June 26, 1995.

A. K. T. Assis is with the Instituto de Física "Gleb Wataghin," Universidade Estadual de Campinas, Unicamp 13083-970 Campinas, São Paulo, Brazil.

M. A. Bueno is with the Instituto de Física "Gleb Wataghin," Universidade Estadual de Campinas, Unicamp 13083-970 Campinas, São Paulo, Brazil.

Publisher Item Identifier S 0018-9464(96)00653-X.

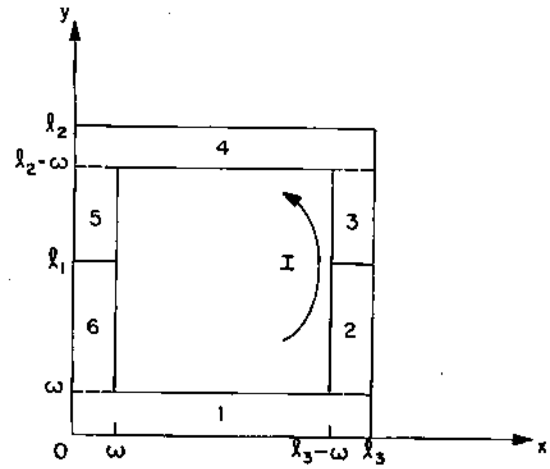

Fig. 1. Rectangular circuit with surface current density. The current is supposed to be uniform over each piece of the circuit, and flows in the following directions: $1 \rightarrow +\hat{x}$, $2 \rightarrow +\hat{y}$, $3 \rightarrow +y$, $4 \rightarrow -\hat{x}$, $5 \rightarrow -\hat{y}$, $6 \rightarrow -\hat{y}$. The bridge consists of pieces 3, 4, and 5; while the support is made of pieces 6, 1, and 2.

Our goal is to calculate the resultant force on the bridge using the forces of Ampère and Grassmann.

As we have surface current elements, we will utilize a generalization of (1) and (2) which avoids the divergences. We only need to replace $Id\vec{l}$ by $\vec{K}da$, where $\vec{K}$ is the surface current density pointing along the current flow and da is the surface element. As we are supposing a uniform current flow, we can write, following Fig. 1: $|\vec{K}| = I/w$, where I is the current crossing the width w of the circuit. In terms of $\vec{K}da$, (1) and (2) can be written as

$$d^4\vec{F}^A_{ji} = -\frac{\mu_0}{4\pi}\frac{\hat{r}}{r^2}\left[2(\vec{K}_i \cdot \vec{K}_j) - 3(\hat{r} \cdot \vec{K}_i)(\hat{r} \cdot \vec{K}_j)\right] da_i da_j, \tag{3}$$

1 on 4, 1 on 3, 2 on 5, and 2 on 4 yields [with (1) and (2)]:

$$\vec{F}^A = \frac{\mu_0 I^2}{2\pi}\left[\ln\frac{l_2}{l_1} - \ln\frac{l_2 - l_1}{l_2} - \ln\frac{l_2 + (l_2^2 + l_3^2)^{1/2}}{l_3} + \frac{(l_2^2 + l_3^2)^{1/2}}{l_2}\right]\hat{y}. \tag{5}$$

$$\vec{F}^G = \frac{\mu_0 I^2}{2\pi}\left[\ln\frac{(l_2 - l_1) + ((l_2 - l_1)^2 + l_3^2)^{1/2}}{l_3} - \ln\frac{l_2 + (l_2^2 + l_3^2)^{1/2}}{l_3} - \ln\frac{l_2 - l_1}{l_2} + \frac{(l_2^2 + l_3^2)^{1/2}}{l_2} - 1\right]\hat{y}. \tag{6}$$

These calculations are straightforward and involve no subtleties.

We must now calculate the force between pieces in contact. Let us calculate the force of 6 on 5 (Fig. 1). It is easily seen that the x component of this resultant force is zero according to either Ampère's force or Grassmann's force. On the other hand, the y component of Ampère's force is given by

$$(F^A_{65})_y = -\frac{\mu_0 I^2}{4\pi w^2}\int_{l_1}^{l_2 - w} dy_5 \int_0^w dx_5 \int_0^w dx_6 \int_w^{l_1} dy_6 \cdot \left[\frac{2(y_5 - y_6)}{[(x_5 - x_6)^2 + (y_5 - y_6)^2]^{3/2}} - \frac{3(y_5 - y_6)^3}{[(x_5 - x_6)^2 + (y_5 - y_6)^2]^{5/2}}\right]. \tag{7}$$

Integrating in the order $y_6 \rightarrow x_6 \rightarrow x_5 \rightarrow y_5$ yields the exact result

$$\vec{F}^A_{65} = \frac{\mu_0 I^2}{4\pi w^2}\Big[2(l_1 - w)(l_2 - l_1 - w) + (l_2 - l_1 - w)(w^2 + (l_2 - l_1 - w)^2)^{1/2} - (l_2 - 2w)(w^2 + (l_2 - 2w)^2)^{1/2} + (l_1 - w)(w^2 + (l_1 - w)^2)^{1/2} + w^2\ln\frac{(l_2 - l_1 - w) + ((l_2 - l_1 - w)^2 + w^2)^{1/2}}{w} - w^2\ln\frac{(l_2 - 2w) + ((l_2 - 2w)^2 + w^2)^{1/2}}{w} + w^2\ln\frac{(l_1 - w) + ((l_1 - w)^2 + w^2)^{1/2}}{w}\Big]\hat{y}. \tag{8}$$

$$d^4\vec{F}^G_{ji} = -\frac{\mu_0}{4\pi}\frac{1}{r^2}\left[(\vec{K}_i \cdot \vec{K}_j)\hat{r} - (\vec{K}_i \cdot \hat{r})\vec{K}_i \cdot \hat{r})\vec{K}_j\right] da_i da_j. \tag{4}$$

We will perform the integrations supposing that the width of the circuit is much smaller than any characteristic length in the system. Relative to Fig. 1, this means that $w << l_1$, $w << l_2$, $w << l_2 - l_1$, and $w << l_3$. This approximation is fulfilled in all the experiments we have seen on this subject. Due to this fact, we can calculate the forces between pieces not in contact using (1) and (2). Adding the forces of 6 on 4 (see Fig. 1), 6 on 3, 1 on 5,

When we calculated the force between portions not in contact using (1) and (2), instead of (3) and (4), we implicitly utilized that $w << l_1$, $w << l_2 - l_1$, and $w << l_3$. For the purpose of consistency, we must now expand (8) utilizing these approximations. This yields, neglecting terms of second order in w/l_1, w/l_2, w/l_3, and above,

$$\vec{F}^A_{65} \cong \frac{\mu_0 I^2}{4\pi}\left[\ln\frac{l_1}{w} + \ln\frac{l_2 - l_1}{l_2} + \ln 2 + \frac{1}{2}\right]\hat{y}. \tag{9}$$

As $\vec{F}^A_{23} = \vec{F}^A_{65}$ (Fig. 1), we can get the resultant force on the bridge according to Ampère's force adding twice

(9) to the linear result (5). This yields

$$F_{SB}^{A} \cong \frac{\mu_0 I^2}{2\pi}\left[\ln\frac{l_2}{w} - \ln\frac{l_2 + (l_2^2 + l_3^2)^{1/2}}{l_3} + \frac{(l_2^2 + l_3^2)^{1/2}}{l_2} + \ln 2 + \frac{1}{2}\right]\hat{y}. \quad (10)$$

Before proceeding, we should observe that we could also arrive at (9) using a different and simpler technique. Instead of integrating exactly and then expanding the final result, we could utilize a series expansion of the integrand. This means, in general, that

$$I \equiv \int_a^{a+w} f(x)\,dx = wf(a) + \frac{w^2}{2!}f'(a) + O(w^3) \quad (11)$$

where $w \ll a$. Using this procedure, we checked the result of (9).

We now calculate $\vec{F}_{65}$ using Grassmann's force. It can be easily shown that this is exactly zero, without any approximations. So, it could be thought that the resultant force on the bridge using Grassmann's expression would be given by (6). But this is not correct. Grassmann's force does not follow the action and reaction law for current elements. This means that the force of the bridge on itself, $\vec{F}_{BB}^{G}$, does not need to be zero (see Section IV). As we shall see, this is indeed the case for Fig. 1. Obviously, this does not happen with Ampère's force because it follows Newton's third law even for current elements, which means that the force of any portion of any circuit (even an open one) on itself will be always zero according to Ampère's expression. This shows that always $\vec{F}_{BB}^{A} = 0$.

Calculating the force of any piece m on itself (Fig. 1, $m = 1, \ldots, 6$) with Grassmann's force, $\vec{F}_{mm}^{G}$, yields zero. Observing also that $(\vec{F}_{43}^{G})_x = -(\vec{F}_{45}^{G})_x$, $(\vec{F}_{53}^{G})_x = -(\vec{F}_{35}^{G})_x$, $(\vec{F}_{34}^{G})_y = (\vec{F}_{54}^{G})_y$, $(\vec{F}_{43}^{G})_y = (\vec{F}_{53}^{G})_y = (\vec{F}_{34}^{G})_x = (\vec{F}_{54}^{G})_x = (\vec{F}_{35}^{G})_y = (\vec{F}_{45}^{G})_y = 0$, yields that the resultant force of the bridge on itself according to Grassmann's force is given by $\vec{F}_{BB}^{G} = 2(\vec{F}_{54}^{G})_y\,\hat{y}$. This was calculated by both methods indicated above, and the approximate result is given by

$$(F_{54}^{G})_y = \frac{\mu_0 I^2}{4\pi w^2}\int_0^{w} dx_5 \int_0^{l_3} dx_4 \int_{l_1}^{l_2-w} dy_5 \cdot \int_{l_2-w}^{l_2} dy_4 \frac{(x_4 - x_5)}{[(x_4 - x_5)^2 + (y_4 - y_5)^2]^{3/2}}$$

$$\cong \frac{\mu_0 I^2}{4\pi}\left[\ln\frac{l_2 - l_1}{w} \cdot - \ln\frac{(l_2 - l_1) + ((l_2 - l_1)^2 + l_3^2)^{1/2}}{l_3} + \ln 2 - \frac{3}{2}\ln(1 + \sqrt{2}) + \frac{\sqrt{2}}{2} + \frac{1}{2}\right]. \quad (12)$$

So, the resultant force of the bridge on itself according to Grassmann's expression is obtained adding (6) to twice

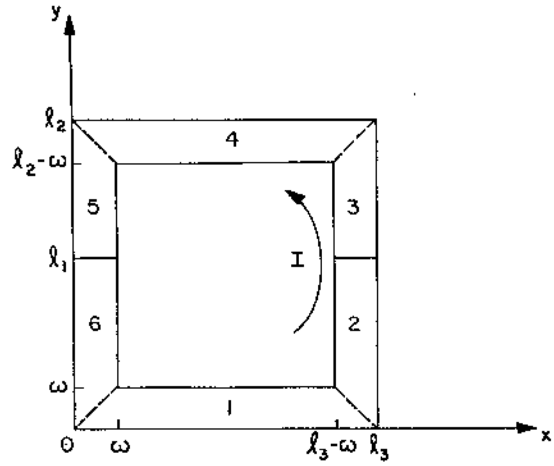

Fig. 2. Second rectangular circuit with surface current density. Bridge: 3, 4, and 5. Support: 6, 1, and 2.

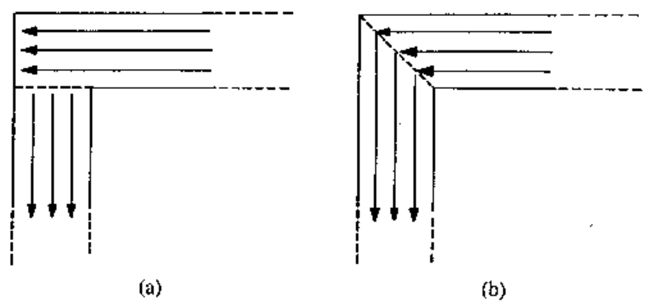

Fig. 3. (a) A representation of the corners of the circuit of Fig. 1. The individual lines of current (indicated by the arrows) are not closed. (b) A representation of the corners of the circuit of Fig. 2. The individual lines of current (indicated by the arrows) are continuous and closed.

(12), namely,

$$F_{SB}^{G} + \vec{F}_{BB}^{G} \cong \frac{\mu_0 I^2}{2\pi}\left[\ln\frac{l_2}{w} - \ln\frac{l_2 + (l_2^2 + l_3^2)^{1/2}}{l_3} + \frac{(l_2^2 + l_3^2)^{1/2}}{l_2} + \ln 2 - \frac{3}{2}\ln(1 + \sqrt{2}) + \frac{\sqrt{2}}{2} - \frac{1}{2}\right]\hat{y}. \quad (13)$$

This is similar to (10), but there is a small difference in the numerical constants. Later on we will discuss this difference.

Our second circuit is that in Fig. 2. The only differences relative to Fig. 1 are in the corners, because now each line of current is closed (Fig. 3). In Fig. 3(a), we have the corners of the circuit of Fig. 1, while in Fig. 3(b), we have the corners of the circuit of Fig. 2. The forces between the pieces not in contact are still given by (5) and (6), as once more we suppose $w \ll l_1$, $w \ll l_2 - l_1$, and $w \ll l_3$. The only aspects which are now changed relative to the previous results are the limits of integration in (7) and (12).

For Ampère's expression in the case of Fig. 2, we utilize (7), but now with y_5 going from l_1 to $l_2 - x_5$, y_6 going from x_6 to l_1, while x_5 and x_6 remain going from 0 to w.

Performing the integration in the order ($y_5 \to y_6 \to x_5 \to x_6$) and expanding the final result, or expanding the integrand from the beginning, yields the same result as (9). This means that the resultant force on the bridge of Fig. 2 is given by (10) according to Ampère's force. For Ampère's force, there was no difference in this approximation for the situations of Figs. 1 and 2.

For Grassmann's expression, we need to calculate $(\vec{F}^G_{54})_y$, and the approximate result is given by

$$(\vec{F}^G_{54})_y = \frac{\mu_0 I^2}{4\pi w^2}\int_0^w dx_5 \int_{l_2-w}^{l_2} dy_4 \int_{l_1}^{l_2-x_5} dy_5 \int_{l_2-y_4}^{y_4+l_3-l_2} dx_4 \frac{(x_4-x_5)}{[(x_4-x_5)^2+(y_4-y_5)^2]^{3/2}}$$
$$\cong \frac{\mu_0 I^2}{4\pi}\left[\ln\frac{l_2-l_1}{w} - \ln\frac{(l_2-l_1)+((l_2-l_1)^2+l_3^2)^{1/2}}{l_3} + \ln 2 + \frac{3}{2}\right] \quad (14)$$

where the limits of the integrals in x_4 and y_4 were obtained considering that: when $y_4 = l_2 - \omega$, x_4 goes from ω to $l_3 - \omega$; when $y_4 = l_2$, x_4 goes from 0 to l_3 (see Fig. 2). That is, let us consider piece 4 of Fig. 2. The straight line between pieces 4 and 5 passes through the points $(0, l_2)$ and $(\omega, l_2 - \omega)$. This means that it is given by $y = -x + l_2$, or $x = l_2 - y$. We obtain, analogously, the equation describing the straight line between pieces 3 and 4, namely: $y = x + (l_2 - l_3)$ or $x = y + l_3 - l_2$. Now, consider in piece 4 a straight line passing through y_4 (where $l_2 - \omega \le y_4 \le y_2$) and parallel to the x-axis. By the previous results, the left and right limits of this line in piece 4 are given by, respectively, $x_4 = l_2 - y_4$ and $x_4 = y_4 + l_3 - l_2$. To cover the whole area of piece 4, the limits of integration are then those given by (14). With the same reasoning, we obtained the limits of integration for x_5 and y_5.

This approximate result was obtained expanding the final exact result (which can be obtained in closed algebraic form integrating in the order $x_4 \to y_5 \to y_4 \to x_5$), or integrating the expanded integrand. Adding twice this result to (6) yields

$$\vec{F}^G_{SB} + \vec{F}^G_{BB} \cong \frac{\mu_0 I^2}{2\pi}\left[\ln\frac{l_2}{w} - \ln\frac{l_2+(l_2^2+l_3^2)^{1/2}}{l_3} + \frac{(l_2^2+l_3^2)^{1/2}}{l_2} + \ln 2 + \frac{1}{2}\right]\hat{y}. \quad (15)$$

This result is exactly the same as (10). This shows that Grassmann's expression predicts exactly the same force on the bridge as Ampère's, even for this single closed circuit! But, in order to arrive at this extremely important result, we needed to take care of two aspects. The first one was to include the force of the bridge on itself when working with Grassmann's force (see Section IV). The second one was to utilize a circuit with only closed and continuous lines of current. Because Wesley did not include these two aspects in his calculations, he wrongly concluded [15] that Ampère's force was the only one compatible with the experiments. Later on we will return to this point.

It can be observed that (10) and (15) do not depend on l_1, which is the height of the bridge. We confirmed this result with the circuit of Fig. 4, which is similar to that of Fig. 2 in the limit $l_1 \to l_2$. In this circuit, the support is given by 1, 2, and 4, while the bridge is only piece 3.

By symmetry, or by direct calculation, it can be shown that the force of each piece on itself is zero according to Ampère and Grassmann's forces, namely, $\vec{F}^A_{mm} = \vec{F}^G_{mm} = 0$, $m = 1, 2, 3$, and 4 (see Fig. 4). It can also be shown that $(\vec{F}^A_{13})_x = (\vec{F}^G_{13})_x = 0$, $(\vec{F}^A_{23})_x = -(\vec{F}^A_{43})_x$, $(\vec{F}^G_{23})_x = -(\vec{F}^G_{43})_x$, $(\vec{F}^A_{23})_y = (\vec{F}^A_{43})_y$, $(\vec{F}^G_{23})_y = (\vec{F}^G_{43})_y$. Therefore, the resultant force on the bridge is given by $\vec{F}^A_{SB} = [2(F^A_{43})_y + (F^A_{13})_y]\hat{y}$, $\vec{F}^G_{SB} = [2(F^G_{43})_y + (F^G_{13})_y]\hat{y}$. The forces $(F^A_{13})_y$ and $(F^G_{13})_y$ can be obtained using (1) and (2) as we are supposing $w \ll l_2$ and $w \ll l_3$. These forces are given by

$$(\vec{F}^A_{13})_y = \frac{\mu_0 I^2}{2\pi}\left[\frac{(l_2^2+l_3^2)^{1/2}}{l_2} - \frac{l_2}{(l_2^2+l_3^2)^{1/2}}\right] \quad (16)$$

$$(\vec{F}^G_{13})_y = \frac{\mu_0 I^2}{2\pi}\left[\frac{(l_2^2+l_3^2)^{1/2}}{l_2} - 1\right]. \quad (17)$$

The forces between parts in contact are given by (expanding the integrand and integrating in the order $x_3 \to y_4 \to y_3 \to x_4$):

$$(F^A_{43})_y = \frac{3\mu_0 I^2}{4\pi w^2}\int_0^w dx_4 \int_{l_2-w}^{l_2} dy_3 \int_{x_4}^{l_2-x_4} dy_4 \int_{l_2-y_3}^{y_3+l_3-l_2} dx_3 \frac{(x_3-x_4)(y_3-y_4)^2}{[(x_3-x_4)^2+(y_3-y_4)^2]^{5/2}}$$
$$\cong \frac{\mu_0 I^2}{4\pi}\left[\ln\frac{l_2}{w} + \frac{l_2}{(l_2^2+l_3^2)^{1/2}} - \ln\frac{l_2+(l_2^2+l_3^2)^{1/2}}{l_3} + \ln 2 + \frac{1}{2}\right] \quad (18)$$

$$(F^G_{43})_y = \frac{\mu_0 I^2}{4\pi w^2}\int_0^w dx_4 \int_{l_2-w}^{l_2} dy_3 \int_{x_4}^{l_2-x_4} dy_4 \int_{l_2-y_3}^{y_3+l_3-l_2} dx_3 \frac{(x_3-x_4)}{[(x_3-x_4)^2+(y_3-y_4)^2]^{3/2}}$$
$$\cong \frac{\mu_0 I^2}{4\pi}\left[\ln\frac{l_2}{w} - \ln\frac{l_2+(l_2^2+l_3^2)^{1/2}}{l_3} + \ln 2 + \frac{3}{2}\right] \quad (19)$$

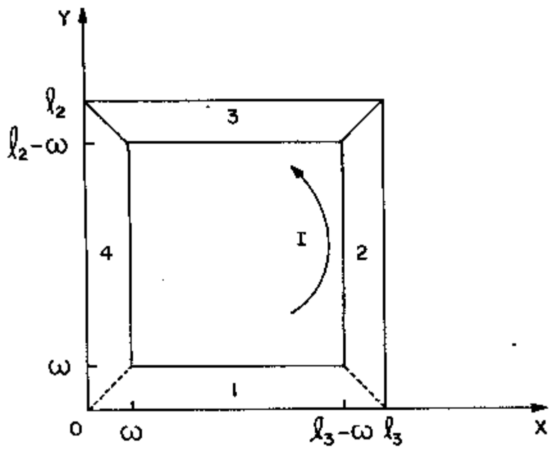

Fig. 4. Third rectangular circuit with surface current density. Bridge: 3. Support: 4, 1, and 2. The corners are like those of Fig. 3(b).

where we have considered that x_3 goes from ω to $l_3 - \omega$ when $y_3 = l_2 - \omega$, and from 0 to l_3 when $y_3 = l_2$ (Fig. 4).

Adding twice (18) with (16) yields (10). Adding twice (19) with (17) yields (10) or (15). Again, Ampère and Grassmann's forces agree exactly on their predictions. We now pass to integrations with volume current elements.

III. Circuit with Volume Current Elements

We now study the circuit represented in Fig. 5. As we have volume current elements, we must replace (1) to (4) by

$$d^6\vec{F}^A_{ji} = -\frac{\mu_0}{4\pi}\frac{\hat{r}}{r^2}\,[2(\vec{J}_i\cdot\vec{J}_j) - 3(\hat{r}\cdot\vec{J}_i)(\hat{r}\cdot\vec{J}_j)]\,dV_i dV_j \tag{20}$$

$$d^6\vec{F}^G_{ji} = -\frac{\mu_0}{4\pi}\frac{1}{r^2}\,[(\vec{J}_i\cdot\vec{J}_j)\hat{r} - (\vec{J}_i\cdot\hat{r})\vec{J}_j]\,dV_i dV_j \tag{21}$$

where $\vec{J}$ is a volume current density and dV a volume element. As we are supposing a uniform flow over each piece of the circuit, we can write $|\vec{J}| = I/w^2$, where I is the current flowing through the cross section of the wire (supposed to be a square with side w). See Fig. 5.

We continue supposing $w << l_1$, $w << l_2 - l_1$ and $w << l_3$. The forces between the parts not in contact are given by (5) and (6). In this case, it can also be shown that $\vec{F}^A_{mm} = \vec{F}^G_{mm} = 0$, $m = 1, \cdots, 6$. Moreover, $(F^A_{15})_x = -(F^A_{13})_x$, $(F^A_{64})_x = -(F^A_{24})_x$, $(F^A_{25})_x = -(F^A_{63})_x$, $(F^A_{15})_y = (F^A_{13})_y$, $(F^A_{64})_y = (F^A_{24})_y$, $(F^A_{25})_y = (F^A_{63})_y$, $(F^A_{14})_x = 0$, $(F^A_{65})_x = -(F^A_{23})_x$, $(F^A_{65})_y = (F^A_{23})_y$, $(F^G_{43})_x = -(F^G_{45})_x$, $(F^G_{53})_x = -(F^G_{35})_x$, $(F^G_{34})_y = (F^G_{54})_y$, $(F^G_{43})_y = (F^G_{53})_y = (F^G_{34})_x = (F^G_{54})_x = (F^G_{35})_y = (F^G_{45})_y = (F^G_{65})_y = 0$.

This means that the resultant forces on the bridge will be given by $\vec{F}^A_{SB} = \vec{F}^A + 2(F^A_{65})_y\hat{y}$ and $\vec{F}^G_{SB} + \vec{F}^G_{BB} = \vec{F}^G + 2(F^G_{54})_y\hat{y}$, where F^A and F^G are given by (5) and (6), respectively. Using the expansion of the integrands we obtain, integrating in the order $y_6 \to y_5 \to x_6 \to x_5 \to z_5 \to z_6$ (see Fig. 5),

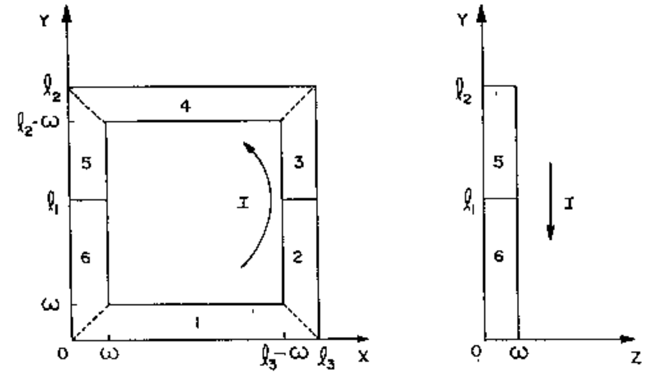

Fig. 5. Rectangular circuit with volume current density. Bridge: 3, 4, and 5. Support: 6, 1, and 2. Uniform direction of the current in each piece: $1 \to +\hat{x}$, $2 \to +\hat{y}$, $3 \to +\hat{y}$, $4 \to -\hat{x}$, $5 \to -\hat{y}$, $6 \to -\hat{y}$.

$$\begin{aligned}(F^A_{65})_y &= \frac{-\mu_0 I^2}{4\pi w^4}\int_0^w dz_6\int_0^w dz_5\int_0^w dx_5 \\ &\cdot \int_0^w dx_6 \int_{l_1}^{l_2 - x_5} dy_5 \int_{x_6}^{l_1} dy_6 \\ &\cdot \left[\frac{2(y_5 - y_6)}{[(x_5 - x_6)^2 + (y_5 - y_6)^2 + (z_5 - z_6)^2]^{3/2}}\right. \\ &\left. - \frac{3(y_5 - y_6)^3}{[(x_5 - x_6)^2 + (y_5 - y_6)^2 + (z_5 - z_6)^2]^{5/2}}\right] \\ &\cong \frac{\mu_0 I^2}{4\pi}\left[\ln\frac{l_2 - l_1}{l_2} + \ln\frac{l_1}{w} + \frac{2}{3}\ln 2 + \frac{13}{12} - \frac{\pi}{3}\right]\end{aligned} \tag{22}$$

where we have considered that y_5 goes from l_1 to l_2 while $x_5 = 0$, and from l_1 to $l_2 - \omega$ while $x_5 = \omega$. We also considered that y_6 goes from 0 to l_1 while $x_6 = 0$, and from ω to l_1 while $x_6 = \omega$.

Twice this value added to (5) yields the resultant force on the bridge according to Ampère's expression

$$\begin{aligned}F^A_{SB} \cong \frac{\mu_0 I^2}{2\pi}\Bigg[\ln\frac{l_2}{w} - \ln\frac{l_2 + (l_2^2 + l_3^2)^{1/2}}{l_3} \\ + \frac{(l_2^2 + l_3^2)^{1/2}}{l_2} + \frac{2}{3}\ln 2 - \frac{\pi}{3} + \frac{13}{12}\Bigg].\end{aligned}$$

We can also get

$$\begin{aligned}(F^G_{54})_y &= \frac{\mu_0 I^2}{4\pi w^4}\int_0^w dz_4\int_0^w dz_5\int_0^w dx_5\int_{l_2 - w}^{l_2} dy_4 \\ &\cdot \int_{l_1}^{l_2 - x_5} dy_5 \int_{l_2 - y_4}^{y_4 + l_3 - l_2} dx_4 \\ &\cdot \left[\frac{(x_4 - x_5)}{[(x_4 - x_5)^2 + (y_4 - y_5)^2 + (z_4 - z_5)^2]^{3/2}}\right] \\ &\cong \frac{\mu_0 I^2}{4\pi}\Bigg[\ln\frac{l_2 - l_1}{l_2} + \ln\frac{l_2}{w} \\ &- \ln\frac{(l_2 - l_1) + ((l_2 - l_1)^2 + l_3^2)^{1/2}}{l_3} \\ &+ \frac{2}{3}\ln 2 + \frac{25}{12} - \frac{\pi}{3}\Bigg].\end{aligned} \tag{24}$$

Twice this value with (6) yields the resultant force on the bridge according to Grassmann's expression

$$\vec{F}_{SB}^{G} + \vec{F}_{BB}^{G} \cong \frac{\mu_0 I^2}{2\pi}\left[\ln\frac{l_2}{w} - \ln\frac{l_2 + (l_2^2 + l_3^2)^{1/2}}{l_3} + \frac{(l_2^2 + l_3^2)^{1/2}}{l_2} + \frac{2}{3}\ln 2 + \frac{13}{12} - \frac{\pi}{3}\right] \quad (25)$$

which is exactly equal to (23).

IV. Discussion and Conclusion

Results (5), (22), and (23) had been obtained by Wesley [15], and we checked them by these independent calculations (he did not consider any circuit with surface current elements). On the other hand, he considered only (6) as the resultant force on the bridge according to Grassmann's expression. But, as we have seen, this is not correct because it is essential to calculate also the force of the bridge on itself, and he did not consider this contribution. Obviously this is not necessary with Ampère's force because, as we have seen, $\vec{F}_{BB}^{A} = 0$ for any kind of bridge. So, his conclusions that Ampère's force is the only one compatible with the experiments of Moyssides and Pappas [6] is not correct. If Ampère's force explains them correctly as he has shown, then Grassmann's force will have exactly the same performance, as we have seen here.

His claim that Grassmann's force predicts bootstrap effect is untenable, when we consider the action of the circuit on itself as a whole. Grassmann's force on a part of the closed circuit results from the action of the whole circuit (support + bridge) on that part. So, when calculating the force on the bridge due to the whole circuit, we also have to take into account the force exerted by the bridge on itself. This was not taken into account in Wesley's work. The division of the circuit in support and bridge is merely for mechanical distinction (these are two mechanically independent parts that could move relative to one another). Electrically it is one closed circuit. For more discussion about this, see [10, p. 4310].

If we speak in terms of the magnetic field, the conclusion of this work is that we must consider the whole magnetic field created by both parts (support + bridge) since we have one electrically closed circuit. Therefore, in this case, since Grassmann's force predicts magnetic fields, the magnetic field of the bridge (along with that of the support) contributes to the net force or motion of the bridge, as a whole.

Moreover, we showed that it is necessary to consider all the lines of current to be continuous and closed, as is the case in all real experiments, in order to obtain correct results. Obviously, the situation of Fig. 2 is idealized. It represents better the experimental situation than the circuit of Fig. 1 with its open lines of current. As we showed in Section II, the correct representation of the lines of current in the circuits will have an important influence on the values predicted by the forces, especially with Grassmann's expression.

Other calculations of which we have knowledge, and which compare Ampère's force and Grassmann's force, are based on numerical integrations [10], [14] or calculating the force via self-inductance [11]. Whenever they considered the action of the closed circuit as a whole and closed lines of current, the result was the same as ours: Ampère and Grassmann's expressions predict the same resultant force on the bridge. As Ampère's force agrees quantitatively with Ampère's bridge experiments, the same happens with Grassmann's [10], [11].

Our work is complementary to that of Moyssides [10]. What he proved numerically in some geometries was proved here algebraically in other situations.

There is another aspect to be touched upon. If we make $w \to 0$, then (10), (13), (15), (18), (19), (23), and (25) will go to infinity. These are the divergences which appear in Ampère's force when we utilize (1) to calculate $\vec{F}_{65}^{A}$ (Figs. 1, 2, 5), or $\vec{F}_{43}^{A}$ (Fig. 4). With Grassmann's force, these divergences appear when we calculate $\vec{F}_{54}^{G}$ (Figs. 1, 2, 5) or $\vec{F}_{43}^{G}$ (Fig. 4) using (2). This shows why we needed to utilize surface and volume current elements to avoid these divergences.

In this work we have showed the equivalence between Ampère and Grassmann's expressions in Ampère's bridge experiment, for the resultant force on the bridge due to the whole circuit. It is not our goal to discuss here the stress distribution caused by these forces. Further research is necessary on the subject [16]–[18].

Acknowledgment

The authors wish to thank FAPESP, FAEP-UNICAMP, and CNPq for financial support over the past few years.

References

[1] A. K. T. Assis, *Weber's Electrodynamics*. Dordrecht: Kluwer, 1994, Ch. 4.

[2] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. New York: Wiley, 1975, pp. 168–173.

[3] R. A. R. Tricker, *Early Electrodynamics—The First Law of Circulation*. Elmsford, NY: Pergamon, 1965, pp. 55–58.

[4] P. Graneau, *Nature*, vol. 295, p. 311, 1982.

[5] P. T. Pappas, *Nuovo Cimento B*, vol. 76, p. 189, 1983.

[6] P. G. Moyssides and P. T. Pappas, *J. Appl. Phys.*, vol. 59, p. 19, 1986.

[7] P. Graneau, *J. Appl. Phys.*, vol. 62, p. 3006, 1987.

[8] ——, *Ampere-Neumann Electrodynamics of Metals*. Nonantum: Hadronic, 1985.

[9] T. E. Phipps and T. E. Phipps, Jr., *Phys. Lett. A*, vol. 146, p. 6, 1990.

[10] P. G. Moyssides, *IEEE Trans. Magn.*, vol. 25, pp. 4298, 4307, 4313, (1989).

[11] V. Peoglos, *J. Phys. D*, vol. 21, p. 1055, 1988.

[12] C. Christodoulides, *J. Phys. A*, vol. 20, p. 2037, 1987.

[13] N. Graneau, *Phys. Lett. A*, vol. 147, p. 92, 1990.

[14] P. Graneau, *J. Appl. Phys.*, vol. 53, p. 6648, 1982.

[15] J. P. Wesley, *Found. Phys. Lett.*, vol. 3, p. 443, 1990.

[16] A. E. Robson and J. P. Sethian, *Amer. J. Phys.*, vol. 60, p. 1111, 1992.

[17] P. Graneau, *Found. Phys. Lett.*, vol. 6, p. 491, 1993.

[18] P. Graneau and N. Graneau, *Galilean Electrod.*, vol. 5, p. 43, 1994.