Deriving Force from Inductance

Marcelo de Almeida Bueno and A. K. T. Assis

***Abstract*—The correctness of calculating the force on current carrying circuits by taking the derivative of the inductance coefficient of the circuits is shown even in the case of a single closed circuit, contrary to the opinion of some authors. This demonstration is illustrated with the force exerted by a cylinder with poloidal current on an infinitesimal strip belonging to the same cylinder. The force is calculated directly from the force expressions and also by differentiating the self-inductance of the cylinder. This result is another proof of the equivalence between Ampère and Grassmann's forces.**

***Index Terms*—Ampère's force, Grassmann's force, inductance.**

IN ORDER TO calculate the force acting on current carrying conductors we can integrate directly the expressions for the force between current elements, for instance Ampère and Grassmann's forces [1]. An alternative way of obtaining these forces is through the inductance coefficient. When we have two closed circuits it is easy to show that the force between them is given by taking the gradient of their mutual magnetic energy [2, pp. 98–102].

Let us give an example of this procedure. In Fig. 1 we have two circular rigid circuits C_1 and C_2 separated by a distance z. Their radius are R_1 and R_2, and they carry currents I_1 and I_2, respectively. They are concentric with the z axis and are located at parallel planes orthogonal to the z axis.

A possible way of calculating the force between them is to integrate directly Ampère or Grassmann's expressions. However, the usual procedure utilized in most cases is not this one. What is employed is the expression

$$\vec{F} = \vec{\nabla} U^N \equiv I_1 I_2 \vec{\nabla} M^N \tag{1}$$

where $U^N = I_1 I_2 M^N$ is the magnetic energy of interaction between the circuits and M^N is Neumann's coefficient of mutual inductance [3]

$$U^N = I_1 I_2 M^N = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{r}_1 \cdot d\vec{r}_2}{r_{12}}. \tag{2}$$

In the approximation $z \gg R_1$ and $z \gg R_2$ we have

$$U^N \approx \frac{\mu_0 I_1 I_2}{2\pi} \frac{(\pi R_1^2)(\pi R_2^2)}{z^3} \tag{3}$$

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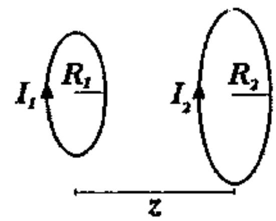

Fig. 1. Two concentric parallel circular circuits of radius R_1 and R_2.

so that the force between the circuits along the z direction, $F = \partial U^N/\partial z$, yields

$$F \approx -\frac{3\mu_0}{2\pi} \frac{(I_1 \pi R_1^2)(I_2 \pi R_2^2)}{z^4} \tag{4}$$

being attractive or repulsive depending on the direction of the currents. The same result is obtained integrating directly Ampère or Grassmann's forces.

As the inductance coefficient has been tabulated for several geometries and is a scalar quantity, this procedure is usually much simpler than integrating directly Ampère or Grassmann's expression.

It is not clear that there is a similar relation between the self-inductance of a single circuit and the net force exerted on a part of itself. Nevertheless, some authors utilize a similar procedure to derive the force on part of specific circuits with the self-inductance (see, for instance, [4] or [5, ch. 23]). The clear advantage of this procedure is that the self-inductances of many circuits have been tabulated for a long time (see, for instance, [5] and [6]) and we only need to perform a single differentiation. The direct integration of Ampère or Grassmann's forces, on the other hand, is usually harder to accomplish.

This procedure needs a clear justification, despite the fact that it yields correct results in most cases, as observed by Wesley when criticizing Peoglos's work [7]. We shall show that this procedure is correct even in the case of a single circuit.

Graneau showed we can derive the force on a current carrying conductor of a single closed circuit by taking the derivative of the self-inductance of this circuit [8, pp. 204–214]. The self-inductance utilized by Graneau is derived from his proposed interaction energy between current elements. The force he obtained with this procedure was Ampère's force between current elements. Recently we have shown that there is a complete equivalence between four known formulae for inductance calculations [3], those of Neumann, Weber, Maxwell, and Graneau. This means that the result obtained by Graneau can be extended directly to the other formulas. In this way, we have completely justified the procedure for deriving force from inductance. Moreover, as the correct inductance formula

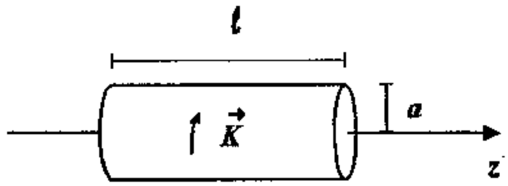

Fig. 2. A cylindrical shell of length ℓ and radius a with poloidal surface current.

in classical electrodynamics is Maxwell's formula [3] and the correct force expression in classical electrodynamics is Grassmann's force [1], from the equivalence between the inductance formulas cited above and the correctness of deriving force from inductance, we also obtain the equivalence between Ampère and Grassmann's forces.

We give an example of this result with the circuit of Fig. 2. We have a cylinder of length ℓ with a circular cross section of radius a, in which flows an uniform poloidal surface current density $\vec{K}$ given by $(I/\ell)\hat{\phi}$, where $\hat{\phi}$ is the unit vector in cylindrical coordinates (ρ, ϕ, z) and I is the total current flowing through the length ℓ. We want to know the force exerted by the whole cylinder on an infinitesimal strip of length ℓ and thickness $ad\phi$ located at $\rho = a, \phi = \pi/2$ and z ranging from zero to ℓ. By symmetry this force is in the $\hat{y}$ direction. Let us first calculate this force by direct integration of Ampère and Grassmann's forces. For the situation in Fig. 2 we have [2, pp. 82, 89, and 112]

$$d\vec{F}^A = \hat{y}\frac{\mu_0 I^2 a^3\, d\phi}{4\pi\ell^2}\int_0^\ell dz_1 \int_0^\ell dz_2 \int_0^{2\pi} d\phi_2 \cdot \left\{\frac{3a^2\cos^2\phi_2(1-\sin\phi_2)}{[(z_1-z_2)^2+2a^2(1-\sin\phi_2)]^{5/2}} - \frac{2\sin\phi_2(1-\sin\phi_2)}{[(z_1-z_2)^2+2a^2(1-\sin\phi_2)]^{3/2}}\right\} \quad (5)$$

$$d\vec{F}^G = \hat{y}\frac{\mu_0 I^2 a^3\, d\phi}{4\pi\ell^2}\int_0^\ell dz_1 \int_0^\ell dz_2 \int_0^{2\pi} d\phi_2 \cdot \frac{1-\sin\phi_2}{[(z_1-z_2)^2+2a^2(1-\sin\phi_2)]^{3/2}}. \quad (6)$$

On evaluating the integrals above, the exact results are

$$d\vec{F}^A = d\vec{F}^G = \hat{y}\frac{\mu_0 I^2 a\, d\phi}{\pi\ell^2} \cdot \left\{(4a^2+\ell^2)^{1/2} \boldsymbol{E}\left[\frac{2a}{(4a^2+\ell^2)^{1/2}}\right] - 2a\right\} \quad (7)$$

where $\boldsymbol{E}$ is the complete elliptic integral of the second kind [9, pp. 907–908]. This shows that Ampère and Grassmann's forces are exactly equal to one another for any value of ℓ/a. If $\ell \gg a$, (7) yields

$$d\vec{F}^A = d\vec{F}^G = \hat{y}\frac{\mu_0 I^2 a\, d\phi}{2\ell} \cdot \left[1 - \frac{4a}{\pi\ell} + \left(\frac{a}{\ell}\right)^2 - \frac{3}{4}\left(\frac{a}{\ell}\right)^4 + O\left(\frac{a}{\ell}\right)^5\right] \quad (8)$$

and if $\ell \ll a$, (7) yields

$$d\vec{F}^A = d\vec{F}^G = \hat{y}\frac{\mu_0 I^2\, d\phi}{4\pi}\left\{\frac{1}{2} + 3\ln 2 - \ln\left(\frac{\ell}{a}\right) + \left(\frac{\ell}{a}\right)^2 \cdot \left[\frac{3 - 12\ln 2 + 4\ln\left(\frac{\ell}{a}\right)}{128}\right] - \left(\frac{\ell}{a}\right)^4 \cdot \left[\frac{3 - 9\ln 2 + 3\ln\left(\frac{\ell}{a}\right)}{1024}\right] + O\left(\frac{\ell}{a}\right)^5\right\}. \quad (9)$$

On the other hand, the self-inductance of the cylinder in Fig. 2 is given by [3]

$$L_{\text{poloidal}} = \frac{\mu_0}{4\pi}\frac{a^2}{\ell^2}\int_0^{2\pi} d\phi_i \int_0^{2\pi} d\phi_j \int_0^\ell dz_i \int_0^\ell dz_j \cdot \left[\left(\frac{1+k}{2}\right)\frac{\cos(\phi_i-\phi_j)}{\{2a^2[1-\cos(\phi_i-\phi_j)]+(z_i-z_j)^2\}^{1/2}} + \left(\frac{1-k}{2}\right)\frac{a^2\sin^2(\phi_i-\phi_j)}{\{2a^2[1-\cos(\phi_i-\phi_j)]+(z_i-z_j)^2\}^{3/2}}\right] = \frac{2\mu_0 a}{3}\left\{p^2\left[\frac{E(q)}{q} - 1\right] - \frac{dE(q)}{dq}\right\} \quad (10)$$

where k is a dimensionless parameter that gives the inductance formulae [3], $p \equiv 2a/\ell, q \equiv p/(1+p^2)^{1/2}$.

Now, if we want the force exerted by the whole cylinder on the infinitesimal strip cited above, we use the expression (see [10, p. 108])

$$d\vec{F} = \hat{y}\frac{I^2\, d\phi}{4\pi}\frac{dL}{da}. \quad (11)$$

On replacing L in (11) as given by (10), we obtain exactly (7), which was calculated integrating directly Ampère and Grassmann's forces.

This shows for the first time, with a concrete example, that we can obtain Ampère or Grassmann's exact forces on part of a closed circuit by taking a simple derivative of the self-inductance of the circuit calculated by the formulas of Neumann, Weber, Maxwell or Graneau.

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