PP. 335 - 341

Weber's Force versus Lorentz's Force

A.K.T. Assis

Abstract
We make a comparison between Weber's force and Lorentz's force. First, we present the historical appearance of these two forces. Then we show their theoretical and conceptual differences. After this we discuss some different predictions of experiments with these two expressions showing how they can be distinguished in the laboratory.

Key words: Weber's force, Lorentz's force, Weber's electrodynamics, Darwin's Lagrangian

1. HISTORICAL BACKGROUND

In this work we compare the forces of Weber and Lorentz. To begin, we present how these expressions appeared historically.[1]

Newton presented his law of universal gravitation in 1687, in the *Principia*. Inspired by Newton's law, Coulomb arrived at his force exerted by charge q_2 on q_1 in 1785. With vectorial notation and in the *Système International* of mksa units it can be written as

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}. \tag{1}$$

In this expression $\varepsilon_0 = 8.85 \times 10^{-12}$ $\mathrm{C^2 \cdot N^{-1} \cdot m^{-2}}$ is the permittivity of free space, the charges q_1 and q_2 are located at $\mathbf{r}_1$ and $\mathbf{r}_2$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $r = |\mathbf{r}|$, and $\hat{\mathbf{r}} = \mathbf{r}/r$.

This force can be derived from a potential energy given by

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{1}{r}. \tag{2}$$

To this end the standard procedure is

$$\mathbf{F} = -\hat{\mathbf{r}}(dU/dr). \tag{3}$$

In 1820 Oersted performed his famous experiments with the magnetized needle near a current-carrying wire. Motivated by these experiments, Ampère performed between 1820 and 1827 a series of classical experiments and arrived at the force exerted by a current element $I_2 d\mathbf{l}_2$ on $I_1 d\mathbf{l}_1$. It can be written as

$$d^2\mathbf{F} = -(\mu_0/4\pi) I_1 I_2 (\hat{\mathbf{r}}/r^2)[2(d\mathbf{l}_1 \cdot d\mathbf{l}_2) - 3(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2)]. \tag{4}$$

In this equation $\mu_0 = 4\pi \times 10^{-7}$ $\mathrm{kg \cdot m \cdot C^{-2}}$ is the vacuum permeability. Integrating this expression, Ampère obtained the force exerted by a closed circuit of arbitrary form carrying a current I_2 on a current element $I_1 d\mathbf{l}_1$ of another circuit, namely,

$$d\mathbf{F} = I_1 d\mathbf{l}_1 \times \left[\frac{\mu_0}{4\pi}\oint_{C_2}\frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}\right]. \tag{5}$$

This shows that the force exerted by a closed circuit on a current element of another circuit is always orthogonal to this element.

If we define the magnetic field of a closed circuit by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\oint_{C_2}\frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \tag{6}$$

then we can write Ampère's force (5) as

$$d\mathbf{F} = I_1 d\mathbf{l}_1 \times \mathbf{B}. \tag{7}$$

In 1845 the mathematician Grassmann wrote a force law between two current elements as

$$d^2\mathbf{F} = I_1 d\mathbf{l}_1 \times d\mathbf{B}_2 = I_1 d\mathbf{l}_1 \times \left[\frac{\mu_0}{4\pi}\frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}\right]. \tag{8}$$

Here the magnetic field $d\mathbf{B}$ of a current element $I_2 d\mathbf{l}_2$ is defined by

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}. \tag{9}$$

Operating the double cross product in Eq. (8) yields

$$d^2\mathbf{F} = -\frac{\mu_0}{4\pi}\frac{I_1 I_2}{r^2}\left[(d\mathbf{l}_1 \cdot d\mathbf{l}_2)\hat{\mathbf{r}} - (d\mathbf{l}_1 \cdot \hat{\mathbf{r}})d\mathbf{l}_2\right]. \tag{10}$$

It should be remarked that Grassmann never performed a single experiment to arrive at these expressions. He created the modern scalar and vector products and wrote these expressions as applications of his mathematics. Obviously, when we integrate Eq. (8) over the closed circuit C_2 acting on a current element of another circuit $I_1 d\mathbf{l}_1$ we obtain the same as Ampère's integrated expression, Eq. (5).

Despite this fact, Ampère never wrote an expression for the magnetic field of a current element or of a closed circuit. He always worked with forces and arrived only at (4) and (5), while never writing or defining expressions (6) and (7).

Wilhelm Weber (1804–1891) presented his force exerted by charge q_2 on q_1 in 1846:

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\left[1 - \frac{\dot{r}^2}{2c^2} + \frac{r\ddot{r}}{c^2}\right] . \tag{11}$$

In this expression $\dot{r} = dr/dt$, $\ddot{r} = d^2r/dt^2$, and c is the ratio of electromagnetic to electrostatic units of charge. This was the first time this electromagnetic constant appeared in physics. Its value was first measured ten years later, in 1856, by Weber and Kohlrausch, who found $c = 3 \times 10^8\ \mathrm{m \cdot s^{-1}}$. In the mksa system it is given by $c = 1/(\mu_0\varepsilon_0)^{1/2}$. Although we have three constants here, only one of them is measured experimentally. Usually we define μ_0 by $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{kg \cdot m \cdot C^{-2}}$, and ε_0 is defined as $\varepsilon_0 = 1/\mu_0 c^2$. In this case only c is measurable.

Alternatively, we could define $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{kg \cdot m \cdot C^{-2}}$ and $c = 1/(\mu_0\varepsilon_0)^{1/2}$ and then only ε_0 would be measurable. In any case only one of these constants is measured, the other two being defined. In the cgs-Gaussian system μ_0 and ε_0 do not appear, only c. This is one of the most important advantages of this system, since it avoids superfluous constants. Incidentally, it should be remarked that the cgs or absolute system owes its existence to Gauss and his collaborator, Wilhelm Weber.

Weber arrived at this force in order to unify electrostatics (Coulomb's force) with electrodynamics [Ampère's force (4)], so that he could derive from a single force both expressions. He also succeeded in deriving Faraday's law of induction (1831) from his force.

Historically he began with Eqs. (1) and (4) and arrived at (11) supposing that the usual conduction current in metallic conductors is due to the motion of charges. The opposite procedure, namely, to begin with Weber's force and arrive at Ampère's force, is easily done. To this end we work with neutral current elements, $I_1 d\mathbf{l}_1$ being composed of dq_{1+} and dq_{1-}, $I_2 d\mathbf{l}_2$ being composed of dq_{2+} and dq_{2-}, such that $dq_{1-} = -dq_{1+}$ and $dq_{2-} = -dq_{2+}$. Then we add the forces of the positive and negative charges of $I_2 d\mathbf{l}_2$ on the positive and negative charges of $I_1 d\mathbf{l}_1$ and utilize that $\mathbf{r}_{1+} = \mathbf{r}_{1-} = \mathbf{r}_1$, $\mathbf{r}_{2+} = \mathbf{r}_{2-} = \mathbf{r}_2$ (infinitesimal current elements) and that

$$\begin{aligned} I_1 d\mathbf{l}_1 &= dq_{1+}(\mathbf{v}_{1+} - \mathbf{v}_{1-}) , \\ I_2 d\mathbf{l}_2 &= dq_{2+}(\mathbf{v}_{2+} - \mathbf{v}_{2-}) . \end{aligned} \tag{12}$$

Two years later, in 1848, he presented his velocity-dependent potential energy from which he could derive his force utilizing Eq. (3), namely,

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{1}{r}\left[1 - \frac{\dot{r}^2}{2c^2}\right] . \tag{13}$$

To see this it is only necessary to remember that as $r = r(t)$ and $\dot{r} = \dot{r}(t) = dr/dt$, then

$$\frac{d\dot{r}^2}{dr} = 2\dot{r}\frac{d\dot{r}}{dr} = 2\dot{r}\frac{d\dot{r}}{dt}\frac{dt}{dr} = 2\ddot{r} . \tag{14}$$

Weber's potential energy was also the first example in physics of a potential energy that depends not only on the distance between the interacting bodies, but also on their velocities. As is well known, this was a very fruitful idea.

Between 1869 and 1871 Weber succeeded in proving that his force law was consistent with the principle of conservation of energy. In particular, he showed that two charges interacting through his force law would keep the value $E = T + U$ constant in time, where U is given by (13), and T is the classical kinetic energy of the system given by

$$T = (m_1 v_1^2)/2 + (m_2 v_2^2)/2. \tag{15}$$

With the charges q_1 and q_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$ moving relative to an inertial frame with velocities $\mathbf{v}_1 = d\mathbf{r}_1/dt$ and $\mathbf{v}_2 = d\mathbf{r}_2/dt$, and with accelerations $\mathbf{a}_1 = d\mathbf{v}_1/dt = d^2\mathbf{r}_1/dt^2$ and $\mathbf{a}_2 = d\mathbf{v}_2/dt = d^2\mathbf{r}_2/dt^2$ we can define

$$\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}, \tag{16}$$

$$\mathbf{v}_{12} = d\mathbf{r}_{12}/dt = \mathbf{v}_1 - \mathbf{v}_2, \tag{17}$$

$$\mathbf{a}_{12} = d\mathbf{v}_{12}/dt = d^2\mathbf{r}_{12}/dt^2 = \mathbf{a}_1 - \mathbf{a}_2. \tag{18}$$

We can then write

$$r = |\mathbf{r}_{12}| = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2}, \tag{19}$$

$$\dot{r} = dr/dt = \hat{\mathbf{r}} \cdot \mathbf{v}_{12}, \tag{20}$$

$$\ddot{r} = d\dot{r}/dt = [\mathbf{v}_{12} \cdot \mathbf{v}_{12} - (\hat{\mathbf{r}} \cdot \mathbf{v}_{12})^2 + \mathbf{r}_{12} \cdot \mathbf{a}_{12}]/r. \tag{21}$$

This transforms Eq. (11) into the form

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\left\{1 + \frac{1}{c^2}\left[\mathbf{v}_{12} \cdot \mathbf{v}_{12} - \frac{3}{2}(\hat{\mathbf{r}} \cdot \mathbf{v}_{12})^2 + \mathbf{r} \cdot \mathbf{a}_{12}\right]\right\}. \tag{22}$$

In 1868 Neumann succeeded in writing Weber's electrodynamics in the Lagrangian formulation. To this end he introduced the Lagrangian energy U_L given by

$$U_L = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{1}{r}\left[1 + \frac{\dot{r}^2}{2c^2}\right] . \tag{23}$$

His Lagrangian was then given by $L = T - U_L$, where T is given by Eq. (15).

Weber's force (22) can be obtained from this Lagrangian by the standard procedure. For instance, the x component of the force exerted by q_2 on q_1 is given by

$$F^x = \frac{d}{dt}\frac{\partial U_L}{\partial \dot{x}_1} - \frac{\partial U_L}{\partial x_1}. \tag{24}$$

Note the sign change in front of $\ddot{r}$ in Eqs. (13) and (23). This also happens in classical electromagnetism, as we will see.

The Hamiltonian H for a two-particle system is given by

$$H = \left[\sum_{k=1}^{k=6} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right] - L. \tag{25}$$

In Weber's electrodynamics it is found to be the conserved energy of the system, namely,

$$H = E = T + U \tag{26}$$

and not $T + U_L$.

We now discuss Lorentz's force $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$. First, let us present how the magnetic component appeared.

The first to write something similar to $\mathbf{v} \times \mathbf{B}$ was J.J. Thomson in 1881, Ref. 2, and Ref. 3, p. 306. Utilizing Maxwell's theory to arrive at the force exerted by a magnet on a charged body, he obtained

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})/2. \tag{27}$$

This is half the present-day value. More interesting is the meaning of the velocity $\mathbf{v}$ to Thomson. He wrote explicitly in this paper that this was the actual velocity of the particle of charge q and explained what he meant by actual velocity: "It should be remarked that what we have for convenience called the actual velocity of the particle is, in fact, the velocity of the particle relative to the medium through which it is moving ..., medium whose magnetic permeability is μ." This shows that to Thomson this was not the velocity of q relative to the observer.

In 1889 Heaviside[4] obtained $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. This is what we accept today for the magnetic force acting on a charge. Although Heaviside did not discuss the meaning of $\mathbf{v}$ in this expression, it is clear from the title of his paper that he accepted the same interpretation as that of Thomson. This is even more evident observing that he was criticizing Thomson's work, but did not say a single word against Thomson's interpretation of $\mathbf{v}$.

H.A. Lorentz (1853–1928) presented his force law in 1895. To our knowledge he never performed a single experiment to arrive at his expression. What were his motivations? Here we present his words in his famous book *The Theory of Electrons*. In square brackets are our words and the modern presentation of some of his formulas (for instance, $[\mathbf{a} \cdot \mathbf{b}]$ is nowadays usually represented by $\mathbf{a} \times \mathbf{b}$). He utilizes the cgs system of units:

However this may be, we must certainly speak of such a thing as the force acting on a charge, or on an electron, on charged matter, whichever appellation you prefer. Now, in accordance with the general principles of Maxwell's theory, we shall consider this force as caused by the state of the ether, and even, since this medium pervades the electrons, as exerted by the ether on all internal points of these particles where there is a charge. If we divide the whole electron into elements of volume, there will be a force acting on each element and determined by the state of the ether existing within it. We shall suppose that this force is proportional to the charge of the element, so that we only want to know the force acting per unit charge. This is what we can now properly call *the electric force*. We shall represent it by $\mathbf{f}$. The formula by which it is determined, and which is the one we still have to add to (17) to (20) [Maxwell's equation's], is as follows:

$$\mathbf{f} = \mathbf{d} + \frac{1}{c}[\mathbf{v} \cdot \mathbf{h}] \qquad \left[\mathbf{f} = \mathbf{d} + \frac{\mathbf{v} \times \mathbf{h}}{c}\right]. \tag{23}$$

Like our former equations, it is got by generalizing the results of electromagnetic experiments. The first term represents the force acting on an electron in an electrostatic field; indeed, in this case, the force per unit charge must be wholly determined by the dielectric displacement. On the other hand, the part of the force expressed by the second term may be derived from the law according to which an element of a wire carrying a current is acted on by a magnetic field with a force perpendicular to itself and the lines of force, an action, which in our units may be represented in vector notation by

$$\mathbf{F} = \frac{s}{c}[\mathbf{i} \cdot \mathbf{h}] \qquad \left[\mathbf{F} = \frac{i d\mathbf{l} \times \mathbf{h}}{c}\right],$$

where $\mathbf{i}$ is the intensity of the current considered as a vector, and s the length of the element. According to the theory of electrons, $\mathbf{F}$ is made up of all the forces with which the field $\mathbf{h}$ acts on the separate electrons moving in the wire. Now, simplifying the question by the assumption of only one kind of moving electrons with equal charges e and a common velocity $\mathbf{v}$, we may write

$$s\mathbf{i} = Ne\mathbf{v},$$

if N is the whole number of these particles in the element s. Hence

$$\mathbf{F} = Ne/c[\mathbf{v} \cdot \mathbf{h}],$$

so that, dividing by Ne, we find for the force per unit charge

$$1/c[\mathbf{v} \cdot \mathbf{h}].$$

As an interesting and simple application of this result, I may mention the explanation it affords of the induction current that is produced in a wire moving across the magnetic lines of force. The two kinds of electrons having the velocity $\mathbf{v}$ of the wire are in this case driven in opposite directions by forces which are determined by our formula.

9. After having been led in one particular case to the existence of the force $\mathbf{d}$, and in another to that of the force $1/c[\mathbf{v}\cdot\mathbf{h}]$, we now combine the two in the way shown in the equation (23), going beyond the direct result of experiments by the assumption that in general the two forces exist at the same time. If, for example, an electron were moving in a space traversed by Hertzian waves, we could calculate the action of the field on it by means of the values of $\mathbf{d}$ and $\mathbf{h}$, such as they are at the point of the field occupied by the particle. [Ref. 5, p. 14.]

We agree with O'Rahilly when he said that this proof of the formula is extremely unsatisfactory, and when he added:

There are two overwhelming objections to this alleged generalization. (1) The two "particular cases" here "combined" are quite incompatible. In the one case we have charges at rest, in the other the charges are moving; they cannot be both stationary and moving. (2) Experiments with a "wire carrying a current" have to do with *neutral* currents, yet the derivation contradicts this neutrality. [Ref. 6, p. 561.]

A very important quest is to know the meaning of the velocity that appears in Lorentz's force. Is it the velocity of the charge q relative to what? Some options are the following: relative to the macroscopic source of the field (namely, a magnet or a current carrying wire), relative to the magnetic field itself, relative to an inertial or arbitrary observer, relative to the laboratory or the Earth, relative to the average motion of the charges (usually electrons) generating the field, and relative to the $\mathbf{B}$ field detector. As we can see from the above quotation ("...force as caused by the state of the ether, and even, since this medium pervades the electrons, as exerted by the ether..."), to Lorentz it was originally the velocity of the charge relative to the ether and not, for instance, relative to the observer. To him, the ether was in a state of absolute rest relative to the frame of fixed stars.

A relevant aspect to take notice is that all of these works of Thomson, Heaviside, and Lorentz were written after Maxwell's death in 1879.

The change of meaning for the velocity that appears in Lorentz's force came with Einstein's paper of 1905 on the special theory of relativity. In it Einstein begins to interpret this velocity as the velocity of charge q relative to an observer or frame of reference. To us, this is the origin of all the confusion that has plagued theoretical physics ever since.

This frequent change in the meaning of $\mathbf{v}$ in a formula so important is very strange, confusing, and unusual in physics. A similar confusion would appear in classical physics with, for instance, a frictional force acting on a projectile on Earth due to air resistance. Supposing it proportional to the velocity $\mathbf{v}$ of the projectile, it can be written as

$$\mathbf{F}_f = -b\mathbf{v}. \tag{28}$$

We always work with $\mathbf{v}$ being the velocity of the projectile relative to the air. For instance, if there is a wind velocity $\mathbf{v}_w$ relative to the surface of the Earth and the projectile is moving relative to the Earth with a velocity $\mathbf{v}_p$, then to apply Newton's laws in this case we would write the damping force as $\mathbf{F}_f = -b(\mathbf{v}_p - \mathbf{v}_w)$. A great confusion would arise if we said that the velocity $\mathbf{v}$ in Eq. (28) should be the velocity of the projectile relative to the Earth, independent of the wind. For the formula to work, we would need in this case to say that the constant b would then depend on the wind, etc. A greater confusion would arise if we said that $\mathbf{v}$ in Eq. (28) were the velocity of the projectile relative to the observer and not to the wind or the Earth. Then for the formula to work, the constant b would need to be a complicated function depending on the observer and on the wind.

This is exactly what happened with the classical electromagnetic force acting on a charge. The simplest thing would be to say that the velocity $\mathbf{v}$ in $q\mathbf{v}\times\mathbf{B}$ was the velocity of q relative to the magnet or wire generating $\mathbf{B}$. But what they said was that this velocity should be meant as the velocity of q relative to a medium of magnetic permeability μ. Then they changed this meaning and said it was the velocity of the charge relative to a very specific medium, the ether. Then they changed it again saying that it was the velocity of the charge relative to the observer. In this case, for the formula to work we need to say that the electric and magnetic fields are a function of the observer, that $\mathbf{E}$ transforms into $\mathbf{B}$ depending on the frame of reference, etc.

Liénard, Wiechert, and Schwarzschild, working in the period 1898 to 1903, obtained expressions for the electric and magnetic fields due to a point charge q_2 located at $\mathbf{r}_2(t)$, moving with instantaneous velocity $\mathbf{v}_2$ and instantaneous acceleration $\mathbf{a}_2$ at time t. Taking into account time retardation, the electric and magnetic fields at another point $\mathbf{r}_1$ on the same time t are given by (after a Taylor expansion of all quantities which depend on the retarded time $t - r/c$ around t and going only up to the second order in $1/c$)

$$\mathbf{E}_2(\mathbf{r}_1) = \frac{q_2}{4\pi\varepsilon_0}\frac{1}{r^2} \times \left[\hat{\mathbf{r}}\left(1 + \frac{\mathbf{v}_2\cdot\mathbf{v}_2}{2c^2} - \frac{3}{2}\frac{(\hat{\mathbf{r}}\cdot\mathbf{v}_2)^2}{c^2} - \frac{\mathbf{r}\cdot\mathbf{a}_2}{2c^2}\right) - \frac{r\mathbf{a}_2}{2c^2}\right], \tag{29}$$

$$\mathbf{B}_2(\mathbf{r}_1) = \frac{q_2}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\mathbf{v}_2\times\hat{\mathbf{r}}}{c^2}. \tag{30}$$

This means that the force on q_1 located at $\mathbf{r}_1(t)$ due to q_2

located at $\mathbf{r}_2(t)$ is given in classical electromagnetism by (up to the second order in $1/c$, inclusive)

$$\mathbf{F} = q_1\left\{\frac{q_2}{4\pi\varepsilon_0}\frac{1}{r^2}\left[\hat{\mathbf{r}}\left(1 + \frac{\mathbf{v}_2\cdot\mathbf{v}_2}{2c^2} - \frac{3}{2}\frac{(\hat{\mathbf{r}}\cdot\mathbf{v}_2)^2}{c^2} - \frac{\mathbf{r}\cdot\mathbf{a}_2}{2c^2}\right) - \frac{r\mathbf{a}_2}{2c^2}\right]\right\} + q_1\mathbf{v}_1 \times \left[\frac{q_2}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{\mathbf{v}_2\times\hat{\mathbf{r}}}{c^2}\right]. \quad (31)$$

What should be stressed here is that in the right-hand side of Eqs. (29) to (31) all the quantities are calculated at the present time t and not at the retarded time $t - r/c$.

In 1920 Darwin obtained a potential energy from which he could derive this expression in the Lagrangian formalism and which is consistent with Einstein's special theory of relativity. It is given by

$$U_L^D = \frac{q_1q_2}{4\pi\varepsilon_0}\frac{1}{r}\left[1 - \frac{\mathbf{v}_1\cdot\mathbf{v}_2 + (\mathbf{v}_1\cdot\hat{\mathbf{r}})(\mathbf{v}_2\cdot\hat{\mathbf{r}})}{2c^2}\right]. \quad (32)$$

The force (31) can be obtained from (32) by (24).

The Hamiltonian, H^D, and the conserved energy of the system, E, are found to be, by (25),

$$H^D = E = E_m + U^D, \quad (33)$$

where

$$U^D = \frac{q_1q_2}{4\pi\varepsilon_0}\frac{1}{r}\left[1 + \frac{\mathbf{v}_1\cdot\mathbf{v}_2 + (\mathbf{v}_1\cdot\hat{\mathbf{r}})(\mathbf{v}_2\cdot\hat{\mathbf{r}})}{2c^2}\right]. \quad (34)$$

Here, E_m may be the classical kinetic energy of the system, (15), or the relative mechanical energy of the two charges if we want a relativistic treatment:

$$E_m = \frac{m_{01}c^2}{(1 - v_1^2/c^2)^{1/2}} + \frac{m_{02}c^2}{(1 - v_2^2/c^2)^{1/2}}. \quad (35)$$

Since we went only up to second order in $1/c$ in Darwin's potential energy, the same should be done in (35).

Observe the sign change in front of the terms in the velocity of the charges that appears in Darwin's Lagrangian potential energy, U_L^D, and in the expression that appears in the conserved energy, U_D in Eq. (33). As we saw previously, this had happened in Weber's electrodynamics [comparing (13) and (23)].

2. CONCEPTUAL AND THEORETICAL DIFFERENCES BETWEEN WEBER'S FORCE AND LORENTZ'S FORCE

The main difference between these two expressions is that Weber's force is completely relational. By the term relational we mean any force that depends only on the relative distance, velocity, acceleration, derivative of the acceleration, etc., between the interacting bodies. That is, any force that depends on r, $\dot{r}$, $\ddot{r}$, d^3r/dt^3, ... Newton's force of universal gravitation and Coulomb's force in electrostatics are of this kind. It was this unique characteristic that made us work with Weber's force in the first place. The classical damping forces are also of this kind, since they depend on the relative velocity between the particle and the medium (air, water, ...) where it is moving. Lorentz's force, on the other hand, depends on the velocity of the test charge relative to the observer and not on the relative velocities between the test charge and the charges with which it is interacting.

We also have that Weber's force is always along the straight line connecting the two charges, no matter how they are moving, while Lorentz's force may have different orientations depending on the velocity and accelerations of the charges.

Another difference is that while Weber's force is symmetrical in the velocities and accelerations, the same does not happen with Lorentz's force. For instance, while in Weber's force (22) there are components of the force depending on the square of the velocities of both charges and also on the accelerations of both charges, the same does not happen with Eq. (31). This last one depends on the square of the velocity of the source charge (the charge that generates the fields), but not on the square of the velocity of the test charge (the charge that is feeling or reacting to the force). It also depends on the acceleration of the source charge, but not on the acceleration of the test charge.

While Weber's force always complies with the principle of action and reaction, the same does not happen in general with Lorentz's force, but only in some very specific cases and symmetrical situations. It is usually argued that this is a positive aspect for Lorentz's force because it implies that the charges do not interact directly with one another, but only with the fields generated by the other charge. The typical picture is that when you accelerate one charge relative to an inertial frame, it will generate an electromagnetic field that will propagate from this charge at light velocity. Only when this field reaches the other charge will it be accelerated by it. So we do not need to have action and reaction. But there is one problem with this picture.

As we saw previously in Eq. (33), there is conservation of energy for this two-charge system even in classical electromagnetism. How can the energy of the charges (not taking into account any energy stored in the electromagnetic field) be conserved in this example, which illustrates the classical picture? The first charge oscillates, then remains at rest, there is a short time interval in which both charges are at rest, and then the second charge is accelerated and oscillates. The formula (33) implies that during all this time at least one of the charges needs to be in motion; they cannot both be stationary, as will happen in this classical picture during the time interval in which the field is traveling from one charge to the other.

Another major difference between Weber's electrodynamics and classical electromagnetism is that we do not need to talk about fields with Weber's force. The only things that matter are the charges, their distance, relative velocities, and relative accelerations. The electromagnetic fields may be introduced in Weber's electrodynamics but only as mathematical constructs without any physical reality. On the other hand, in classical electromagnetism the fields are all important. They are the inter - mediate agents between the charges, they carry energy and moment, after their generation they exist independent of the charges, etc.

3. DIFFERENT EXPERIMENTAL PREDICTIONS

Can these two force laws be distinguished experimentally? The answer is yes. Here we discuss some possibilities. First, let us talk about forces between currents. With Weber's force we can derive Ampère's force between current elements, but not Grassmann's force. On the other hand, with Lorentz's force we can derive only Grassmann's force, but not Ampère's force. If these two forces (Ampère and Grassmann) can be distinguished experimentally, then we can eliminate one of these expressions (Weber or Lorentz), whichever is against the experimental findings. We are not going to discuss any of these experiments, but refer the reader to some important literature regarding this subject, which can be found, for instance, in Refs. 7 to 11.

Related to this topic we can mention another interesting aspect. Helmholtz presented a general expression for the electromagnetic energy between two current elements.[12-14] It is given by

$$d^2U = \frac{\mu_0}{4\pi} I_1 I_2 \left[\frac{1+k}{2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} + \frac{1-k}{2} \frac{(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\mathbf{r} \cdot d\mathbf{l}_2)}{r} \right] . \tag{36}$$

According to him, with $k = 0$ we have classical electromagnetism (Maxwell's theory), with $k = -1$ we have Weber's electrodynamics, and with $k = 1$ we have the result proposed by Neumann.

We can show that this is indeed the case calculating from Eqs. (13) and (34) the energy of interaction between two current elements. Adding the energy of dq_{2+} interacting with dq_{1+} and dq_{1-}, with the energy of dq_{2-} interacting with dq_{1+} and dq_{1-} yields, with Eq. (12),

$$d^2U = \frac{\mu_0}{4\pi} I_1 I_2 \frac{(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2)}{r} , \tag{37}$$

$$d^2U = \frac{\mu_0}{4\pi} I_1 I_2 \frac{(d\mathbf{l}_1 \cdot d\mathbf{l}_2) + (\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2)}{2r} . \tag{38}$$

These expressions are the electromagnetic energies of interaction of two current elements in Weber's electrodynamics (first expression) and in classical electromagnetism (second expression).

In order to obtain the energy of interaction of two different closed circuits C_1 and C_2 we utilize that

$$M = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2)}{r} , \tag{39}$$

where M is called the coefficient of mutual induction between these two circuits (introduced by Neumann in 1845).

It is then easily seen that for this case of two different closed circuits the energy of the two circuits will be given in classical electromagnetism and in Weber's electrodynamics by

$$U = I_1 I_2 M. \tag{40}$$

This result would also arise from Helmholtz's general expression (36), since the integrated result is independent of k.

For part of a closed circuit interacting with the remainder of this circuit we may have different values according to Weber's electrodynamics or classical electromagnetism.

We now discuss a specific difference between Weber's force and Lorentz's force. In principle, it may be put to experimental test so that in the near future we can decide between these two force laws directly, and not indirectly as it would happen if we could decide between Ampère's force and Grassmann's force. We discussed this new situation at length in two recent papers.[15,16]

Suppose we have a spherical shell made of a dielectric material, charged uniformly with a total charge Q. The shell has a radius R and is spinning relative to the Earth with an angular velocity $\omega(t)$. Putting the center of the shell (which is always at rest relative to the Earth) at the origin of a coordinate system, we can calculate the force on a test particle q located inside the shell at $\mathbf{r}(t)$ ($r < R$) and moving with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ relative to the Earth. With Weber's force (22) we get

$$\mathbf{F} = \frac{qQ}{12\pi\varepsilon_0 Rc^2} \left[\mathbf{a} + \omega \times (\omega \times \mathbf{r}) + 2\mathbf{v} \times \omega + \mathbf{r} \times \frac{d\omega}{dt} \right] . \tag{41}$$

Let us concentrate on the situation in which $d\omega/dt = 0$, so that the last term on this equation can be neglected.

Classically this last situation yields only a constant electrical potential inside the shell, so that it generates no electric field. On the other hand, it generates a uniform and constant magnetic field anywhere inside the shell given by (Ref. 17, pp. 61, 250; Ref. 18, pp. 229, 289)

$$\mathbf{B} = \mu_0 Q\omega / 6\pi R. \tag{42}$$

This means that the net force on the charge according to Lorentz's force is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = (qQ\mathbf{v} \times \omega)/6\pi\varepsilon_0 Rc^2. \tag{43}$$

Comparing Eqs. (41) with $d\omega/dt = 0$ and (43) we can see that in this case Lorentz's force is only one component of Weber's force, namely, the equivalent to an electrical Coriolis force. On the other hand, there are two components of Weber's force that have no equivalent in classical electromagnetism: the centrifugal electrical force $m_w \omega \times (\omega \times \mathbf{r})$ and the inertial electrical force $m_w\mathbf{a}$, where we defined what we call Weber's inertial mass m_w by $m_w = qQ/12\pi\varepsilon_0 Rc^2$.

These are great differences between the two forces and could be put under experimental scrutiny. Even for a stationary and nonspinning shell the two laws predict different results. While the shell will not exert any force whatsoever on any internal particle according to classical electromagnetism, Weber's expression predicts a force due to the shell on any internal charged particle that is accelerated by any means (by other charged bodies, by magnets, springs, etc.). The effect of this term is equivalent to a change in the inertial mass of the test charge which would depend on the charge and radius of the shell, or on its electric potential $V = Q/4\pi\varepsilon_0 R$. There is no equivalent of this effect in classical electromagnetism, since Lorentz's force does not depend on the acceleration of the test charge. According to Weber's electrodynamics and Newtonian mechanics, the test charge would behave inside a charged spherical shell as if it had an effective inertial mass given by $m - m_w$, where m is its usual inertial mass.

What is the order of magnitude of this effect? If we take an electron as our test particle, we could make its effective inertial mass double its usual value of 9×10^{-31} kg or make it go to zero accelerating it inside a spherical shell charged to a potential of $V = \pm 1.5 \times 10^6$ V.

Will it happen? We believe Weber's force will be vindicated by experiments of this kind, but only nature can give the final answer.

Further discussion on the topics presented in this paper can be found in the recent book, *Weber's Electrodynamics*.[19]

Acknowledgment
The author wishes to thank CNPq, FAPESP and FAEP (Brazil) for financial support during the past few years.

Received 3 October 1994.

Résumé
Nous faisons une comparaison entre les forces de Weber et de Lorentz. Tout d'abord, nous discutons de leur apparition du point de vue historique. Puis, nous montrons leurs différences théoriques et conceptuelles. Finalement, nous discutons de prédictions d'expériences en montrant comment on peut distinguer ces deux expressions en laboratoire.

References

1. A.K.T. Assis, Can. J. Phys. **70**, 330 (1992).
2. J.J. Thomson, Philos. Mag. **11**, 229 (1881).
3. E.T. Whittaker, *A History of the Theories of Aether and Electricity* (Humanities Press, NY, 1973), Vol. 1.
4. O. Heaviside, Philos. Mag. **27**, 324 (1889).
5. H.A. Lorentz, *The Theory of Electrons — And Its Applications to the Phenomena of Light and Radiant Heat* (Teubner, Leipzig, 1915). Reprinted in *Selected Works of H.A. Lorentz*, edited by N.J. Nersessian (Palm Publications, Nieuwerkerk, 1987), Vol. 5.
6. A. O'Rahilly, *Electromagnetic Theory — A Critical Examination of Fundamentals* (Dover, NY, 1965).
7. P. Graneau, *Ampère-Neumann Electrodynamics of Metals* (Hadronic Press, Nonantum, 1985).
8. P. Graneau and N. Graneau, *Newton Versus Einstein — How Matter Interacts With Matter* (Carlton Press, NY, 1993).
9. T.E. Phipps, Jr., Phys. Essays **3**, 198 (1990).
10. *Idem*, Apeiron **17**, 1 (1993).
11. T.E. Phipps and T.E. Phipps, Jr., Phys. Lett. A **146**, 6 (1990).
12. A.E. Woodruff, Isis **59**, 300 (1968).
13. M.N. Wise, in *Conceptions of Ether — Studies in the History of Ether Theories 1740–1900*, edited by G.N. Cantor and M.J.S. Hodge (Cambridge University Press, Cambridge, 1981), p. 269.
14. T. Archibald, Archives Int. d'Hist. Sci. **39**, 276 (1989).
15. A.K.T. Assis, Commun. Theoret. Phys. **18**, 475 (1992).
16. *Idem*, J. Phys. Soc. Japan **62**, 1418 (1993).
17. V.V. Batygin and I.N. Toptygin, *Problems in Electrodynamics* (Academic, London, 1964).
18. D.J. Griffiths, *Introduction to Electrodynamics*, 2nd ed. (Prentice Hall, Englewood Cliffs, 1989).
19. A.K.T. Assis, *Weber's Electrodynamics* (Kluwer Academic, Dordrecht, 1994), Vol. 66 of the series *Fundamental Theories of Physics*.

A.K.T. Assis
Instituto de Física "Gleb Wataghin"
Universidade Estadual de Campinas - Unicamp
13083-970 Campinas, São Paulo, Brazil

Also with:
Department of Applied Mathematics, IMECC
Universidade Estadual de Campinas
13081-970 Campinas, São Paulo, Brazil

e-mail: assis@ifi.unicamp.br